Standards Unit

Improving learning in mathematics: challenges and strategies

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Contents

Introduction

For the past two years, the Standards Unit Mathematics Team has been working with over 200 teachers[1] across 40 organisations nationwide to develop and improve the teaching and learning of mathematics in the learning and skills sector. The approaches and materials contained in this resource are the result of this work.

In providing these resources, we are not attempting to offer a complete course; rather, we aim to exemplify effective and enjoyable ways of teaching and learning mathematics. These approaches are based on earlier research and have now been tried and tested in a wide variety of contexts, including colleges, schools, prisons and work-based learning providers. Throughout, we have collected systematic feedback from teachers and learners. This has led to much rewriting and refinement of our initial ideas. Overwhelmingly, the feedback has been enthusiastic and positive. We are now at a stage where we would like to share these ideas with a wider audience.

The approaches suggested in these materials are challenging. We find that the activities cause us to rethink some of our fundamental assumptions and beliefs about teaching and learning. Our expectations of, and respect for, learners have grown considerably as we have begun to ask more demanding questions, set more complex and engaging tasks and asked learners to adopt more active roles in their learning.

[1] In this publication we use the term 'teachers' as a convenient term to refer to all those teachers, trainers, lecturers and tutors involved in educating post-16 learners.

These activities may be new to many teachers and so, for the sake of clarity, we have been quite specific in describing them. Please forgive us, therefore, if at times we appear to be stating the obvious. We recognise that all teachers have their own individual styles and ways of working and will want to adapt these resources for use in very different contexts. This is wholly natural and to be encouraged. Considerable research and thought have gone into developing these activities, however, and the approaches described here work well in most situations. We therefore strongly suggest you try using them 'as written' at least for the first time. Those who persist in using the activities often report their surprise and delight at the gradual improvement in the motivation, confidence and learning of their learners.

This book attempts to explain why we have designed the activities as we have, and shares our thinking about the management and organisation of the activities. Some readers might like to look at a sample of the materials and to view sections of the DVD-ROM to gain a 'flavour' of the activities before reading this book.

We hope you find it useful and a stimulus to your own creativity.

1 • The purpose of these resources

"In many of the mathematics lessons where learning is unsatisfactory one or more of the following characteristics are evident:

- students are given low-level tasks which are mechanistic and can be completed by imitating a routine or procedure without any depth of thought
- students are mainly receivers of information, and have little opportunity for more direct participation in the lesson and the exploration of different approaches
- insufficient time is allowed for students to develop their understanding of the mathematical concepts being taught
- students have too little time to explain their reasoning and consider the merits of alternative approaches." [16][2]

1.1 From 'passive' to 'active' learning

Teaching does not always result in learning. This may seem self-evident but, as inspection reports tell us, most teachers of mathematics continue to use 'chalk and talk', while learners continue to adopt passive learning strategies. In our own survey of about 750 learners of mathematics from over 30 FE and sixth-form colleges, learners described their most frequent behaviours in the following ways:

"I listen while the teacher explains."
"I copy down the method from the board or textbook."
"I only do questions I am told to do."
"I work on my own."
"I try to follow all the steps of a lesson."
"I do easy problems first to increase my confidence."
"I copy out questions before doing them."
"I practise the same method repeatedly on many questions."

For these learners, mathematics is something that is 'done to them', rather than being a creative, stimulating subject to explore. It has become a collection of isolated procedures and techniques to learn by rote, rather than an interconnected network of interesting and

[2] Numbers in square brackets indicate reference to sources. See pages 61 ff.

powerful ideas to actively explore, discuss, debate and gradually come to understand.

Our first aim in designing this resource is to make mathematics teaching more effective by challenging learners to become more active participants. We want them to engage in discussing and explaining their ideas, challenging and teaching one another, creating and solving each other's questions and working collaboratively to share their results. They not only improve in their mathematics; they also become more confident and effective learners.

1.2 From 'transmission' to 'challenging' teaching

Traditional teaching methods are sometimes called 'transmission' approaches; methods are explained to learners one step at a time. Teachers only question learners in order to lead them in a particular direction or to check they are following the taught procedure. Learners are expected to achieve fluency through practising these methods on lists of graded exercises.

"This active approach is developing learners' desire to learn and interest in maths. By the end of the session there was a real interest in fractions and a will to understand better."
Jim Ferguson
Peterborough Regional College

Transmission approaches can appear superficially effective when short-term recall is required, but they are less effective for longer-term learning, because they:

- encourage the rote memorising of disconnected rules, which are often misapplied and quickly forgotten;
- take no account of learners' prior knowledge (and misunderstandings);
- encourage a passive attitude among learners, who feel that they have nothing to contribute. "Just tell me what to do";
- encourage learners to measure their success by how many questions they have done, rather than by what they have understood.

The model of learning proposed by these resources is different. Our view of teaching and learning is 'connectionist' [2, 5, 22], because it emphasises the interconnected nature of the subject, and it is 'challenging' [26] because it confronts common difficulties through careful explanation rather than attempts to avoid them. It begins from a different set of beliefs about mathematics, learning and teaching, summarised in Figure 1.

Our model of learning should not be confused with that of 'discovery' teaching, where the teacher simply presents tasks and expects learners to explore and discover the ideas for themselves. We see the teacher as having a much more pro-active role than this.

Figure 1

A 'Transmission' view		'Connected', 'challenging' view
A given body of knowledge and standard procedures that has to be 'covered'.	**Mathematics is …**	An interconnected body of ideas and reasoning processes.
An individual activity based on watching, listening and imitating until fluency is attained.	**Learning is …**	A collaborative activity in which learners are challenged and arrive at understanding through discussion.
Structuring a linear curriculum for learners. Giving explanations before problems. Checking that these have been understood through practice exercises. Correcting misunderstandings.	**Teaching is …**	Exploring meaning and connections through non-linear dialogue between teacher and learners. Presenting problems before offering explanations. Making misunderstandings explicit and learning from them.

> "I found students to be highly motivated by the materials. So much so that they have asked to do more of this. This is much better than leading from the front."
> **Hugh Campbell**
> *Derby College*

> "In the prison service this approach has taken the demon out of mathematics. They come hating mathematics and now enjoy it."
> **Jackie Napier**
> *HMP Drake Hall*

The teacher's role in our model is to:

- assess learners and make constructive use of prior knowledge;
- choose appropriate challenges for learners;
- make the purposes of activities clear;
- help learners to see how they should work together in profitable ways;
- encourage learners to explore and exchange ideas in an unhurried, reflective atmosphere;
- encourage the discussion of alternative methods and understandings, examining their strengths and weaknesses;
- remove the 'fear of failure' by welcoming mistakes as learning opportunities rather than problems to avoid;
- challenge learners through effective, probing questions;
- manage small group and whole group discussions;
- draw out the important ideas in each session;
- help learners to make connections between their ideas.

2 • Some underlying principles

> "Lessons are now far more enjoyable for students. I would like to adapt the materials for all teaching sessions."
> **Helen Johnson**
> *Solihull College*

These resources, we believe, are designed to encourage 'best practice' – research-based and with proven effectiveness. The activities have undergone several rounds of observed trials in a variety of learning contexts: colleges, schools, prisons and work-based learning environments. The feedback has been used to refine the materials. We have also built on the research and development work of earlier research projects in further education [22, 24].

The resources by themselves, however, do not guarantee effective teaching. This is entirely dependent on how they are used. When using them, therefore, we suggest that you try to implement the following principles that should underlie all good teaching.

(i) Build on the knowledge learners bring to sessions

> "I did not anticipate the levels of skill and understanding that I found."
> **Marian Ebrey**
> *HMP Hewell Grange*

Effective teaching assumes that learners do not arrive at sessions as 'blank slates', but as actively thinking people with a wide variety of skills and conceptions. Research shows that teaching is more effective when it assesses and uses prior learning to adapt to the needs of learners [9]. This prior learning may be uncovered through any activity that offers learners opportunities to express their understanding. It does not require more testing. For example, it can take the form of a single written question given at the beginning of a session to set the agenda for that session and elicit a range of explanations. These responses may then be used as a starting point for discussion.

(ii) Expose and discuss common misconceptions

Research has shown that teaching becomes more effective when common mistakes and misconceptions are systematically exposed, challenged and discussed [3]. The sessions described here typically begin with a challenge that exposes learners' existing ways of thinking. Cognitive conflicts occur when the learner recognises inconsistencies between existing beliefs and observed events. This happens, for example, when a learner completes a task using more than one method and arrives at conflicting answers. Activities are carefully designed so that such conflicts are likely to occur. Research has shown that such conflicts, when resolved through reflective discussion, lead to more permanent learning than conventional, incremental teaching methods, which seek to avoid learners making 'mistakes'.

> "I always asked a lot of questions and thought they were really helpful. I now realise these sometimes closed discussion down or cut them off. Now I step back and let the discussion flow more. This is very hard to do."
> **Mandy Cave**
> *Carmel College*

(iii) Develop effective questioning

There have been many studies of teachers' questioning. Typically, most questions are low-level, testing the ability of learners to recall facts and procedures. Such questions are also called 'closed', meaning that they permit just one single correct response. Fewer questions promote higher-level reflective thinking, such as the ability to apply, synthesise or explain. Such questions are called 'open' because they invite a range of responses. The research evidence shows that a variety of lower-level and higher-level open questions is much more beneficial than a continuous diet of closed recall questions.

A second finding is the importance of allowing time for learners to think before offering help or moving on to ask a second learner. Studies have shown that many teachers wait for less than one second. Longer 'wait times' are associated with significantly improved achievement [3].

(iv) Use cooperative small group work

Many learners think that learning mathematics is a private activity. They frequently enter post-16 education under-confident and reluctant to discuss difficulties. It is therefore essential that a supportive and encouraging atmosphere is created in the learning environment. It is the teacher's responsibility to ensure that everyone feels able to participate in discussions and this is often easier in small group situations. It is interesting to consider why small group activities are used less often in mathematics than in other subject areas, where they are commonly used to good effect. One possible reason might be the lack of suitable resources. We hope that this resource will help to fulfil this need.

There is now general agreement in research that cooperative small group work has positive effects on learning, but that this is dependent on the existence of shared goals for the group and individual accountability for the attainment of these goals. It has also been seen to have a positive effect on social skills and self-esteem [3].

(v) Emphasise methods rather than answers

Often we find that learners focus more on obtaining a correct answer than on learning a powerful method. They often see their task as 'getting through' an exercise rather than working on an idea. Completion is seen as more important than comprehension. In these resources, we do not concern ourselves with whether or not learners complete every task, but instead we try to increase their power to explain and use mathematical ideas. Learners may work on fewer problems than in conventional texts, but they come to understand them more deeply as they tackle them using more than one method.

(vi) Use rich collaborative tasks

Rich tasks:

- are accessible and extendable;
- allow learners to make decisions;
- involve learners in testing, proving, explaining, reflecting, interpreting;
- promote discussion and communication;
- encourage originality and invention;
- encourage 'what if?' and 'what if not?' questions;
- are enjoyable and contain the opportunity for surprise. [1]

Textbooks often assume that we should begin topics by solving simple questions and then gradually move towards more complex questions. While this may appear natural, we find that learners tend to solve simple questions by intuitive methods that do not generalise to more complex problems. When the teacher insists that they use more generalisable methods, learners do not understand why they should do so when intuitive methods work so well. Simple tasks do not motivate a **need** to learn.

Rich tasks also allow **all** learners to find something challenging and at an appropriate level to work on.

“I used the teaching methods in other lessons. This proved very effective. Learners who find maths difficult benefit from their peers within the groups.”
Steve Woodward

“Even those who sit back were drawn into the activities.”
Sue Sealey
S&B Training Ltd.

(vii) Create connections between mathematical topics

A common complaint of teachers is that learners find it difficult to transfer what they learn to similar situations. Learning appears compartmentalised and closely related concepts and notations (such as division, fraction and ratio) remain unconnected in learners’ minds. In this resource, we have therefore included ‘linking activities’ that are particularly designed to draw out connections across mathematical topics. The index refers to sessions as ‘mostly number’ or ‘mostly algebra’ in order to reflect these connections.

(viii) Use technology in appropriate ways

While new technologies have transformed our lives in many ways, they have had less impact inside most mathematics classrooms. They do offer us the opportunity to present mathematical concepts in dynamic, visually exciting ways that engage and motivate learners. In the sessions that follow, we have sought to illustrate some of this potential through the provision of a few computer ‘applets’; these are small pieces of purpose-built software that are designed to be very easy to use.

3 • Getting started

The activities in this resource attempt to make thinking more visible. When learners work quietly and individually on pencil and paper tasks, it is difficult for the teacher to monitor what they are doing. Learning becomes a private affair. During a busy session there is insufficient time for the teacher to go round and spend time with each individual, diagnosing their difficulties and suggesting ways forward. Often the teacher only discovers what learners might have been thinking some hours later, when the work is marked. In the collaborative activities we shall describe, however, thinking becomes more public and open to scrutiny. The visible products of learning are often larger (e.g. posters), and the teacher is more able to monitor the work, spot errors and respond in flexible, appropriate ways. This is the essence of good, formative assessment.[3]

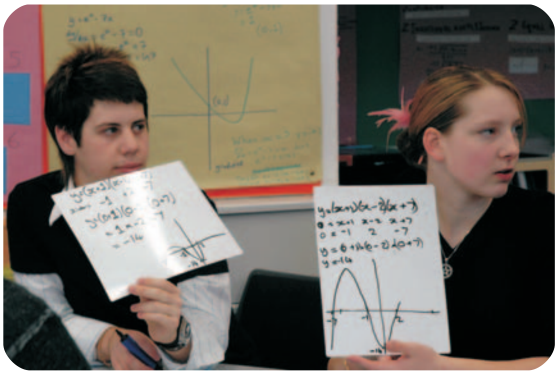
3.1 Arranging the room to facilitate discussion

We found that the quality of discussion was much enhanced when the room was rearranged to facilitate this. Clearly this was not always possible as, for example, when sessions were timetabled in science labs. It is clearly unhelpful to have learners sitting in rows. People find discussion impossible when all they can see are the backs of people's heads.

'Horseshoe' (or 'double horseshoe') arrangements may assist whole group teacher–learner and learner–learner discussions. Some learners will need to move round, however, when working in small groups. When tables are arranged in blocks, groups of learners can gather round and discuss easily. We would recommend this arrangement particularly when card matching activities are used, as these need a considerable amount of table space. Learners may also move from one block to another when considering the work produced by other groups.

[3] Assessment may serve a variety of different purposes, both summative (for grading, selection and certification) and formative (identifying what has been learned or not learned to inform future teaching). In this section we are using assessment in its formative sense.

3.2 Questioning with mini-whiteboards



Mini-whiteboards are becoming increasingly common in education, largely as a result of the Skills for Life strategy with adult learners and the National Mathematics Strategy in schools. We have found them an indispensable resource for the following reasons.

- When learners hold their ideas up to the teacher, it is possible to see at a glance what every learner thinks.
- During whole group discussions, they allow the teacher to ask new kinds of question (typically beginning: 'Show me . . .').
- They allow learners to simultaneously present a range of written and/or drawn responses to the teacher and to each other.
- They encourage learners to use private, rough working that can be quickly erased.

Examples of a range of 'Show me . . .' questions are given below. Notice that most of these are 'open questions' that allow a range of responses. It is worth encouraging a range of such responses with instructions like: "Show me a really different example"; "Show me a complicated example"; "Show me an example that is different from everyone else on your table".

Typical 'show me' open questions

Show me:

- Two fractions that add to 1 . . . Now show me a different pair.
- A number between $\frac{1}{3}$ and $\frac{1}{4}$ . . . Now between $\frac{1}{3}$ and $\frac{2}{7}$.
- The equation of a straight line that passes through (2,1) . . . and another.
- A quadratic equation with a minimum at (2,1) . . . and another.
- A quadrilateral with two lines of symmetry.
- A quadrilateral with a rotational symmetry but no lines of symmetry.
- A hexagon with two reflex angles . . . A pentagon with four right angles.
- A shape with an area of 12 square units . . . and a perimeter of 16 units.
- A set of 5 numbers with a range of 6 . . . and a mean of 10 . . . and a median of 9.

Some teachers find it helpful to write a few of the learners' answers (anonymously) on the board for discussion, both correct and incorrect. On the board, responses can become 'detached' from learners and they feel less threatened when these are criticised by others. This encourages risk taking. Some teachers also introduced answers that were not given by learners but which brought out some particular learning point that they wished to emphasise.

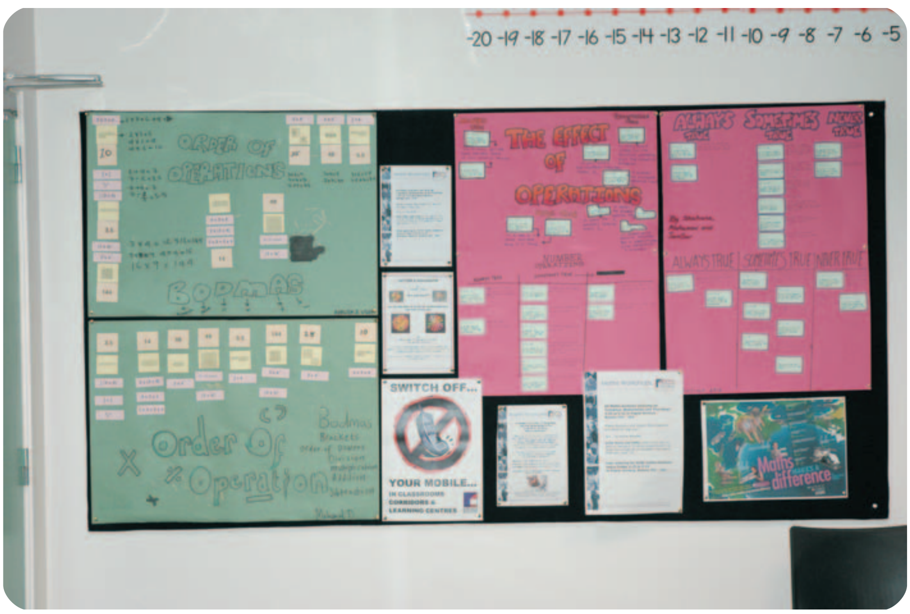
After asking an open question, it is important to welcome and encourage answers, but not pass immediate judgement on them. "Thanks, that is a really interesting answer. Does anyone have something different?" will generate discussion, whereas "That is a really good answer." will inhibit discussion, because learners with alternative ideas tend to remain silent. Judgements should thus be reserved for the end of a discussion.

3.3 Using posters to stimulate thinking

In primary and secondary schools, posters are often used to display the finished, polished work of learners. In our own work, however, we have used them to promote collaborative thinking for formative assessment. This is a very different use. The posters are not produced at the end of the learning activity; they **are** the learning activity and they show all the thinking that has taken place, 'warts and all'.

"Learners work together to create posters that connect ideas together. Learners love working together using sugar paper and lots of different coloured big felt tips. I do not let them do rough work first – the poster is a record of their solving of the problem and their thought processes. It is not a 'perfect' copy of what they have done previously.

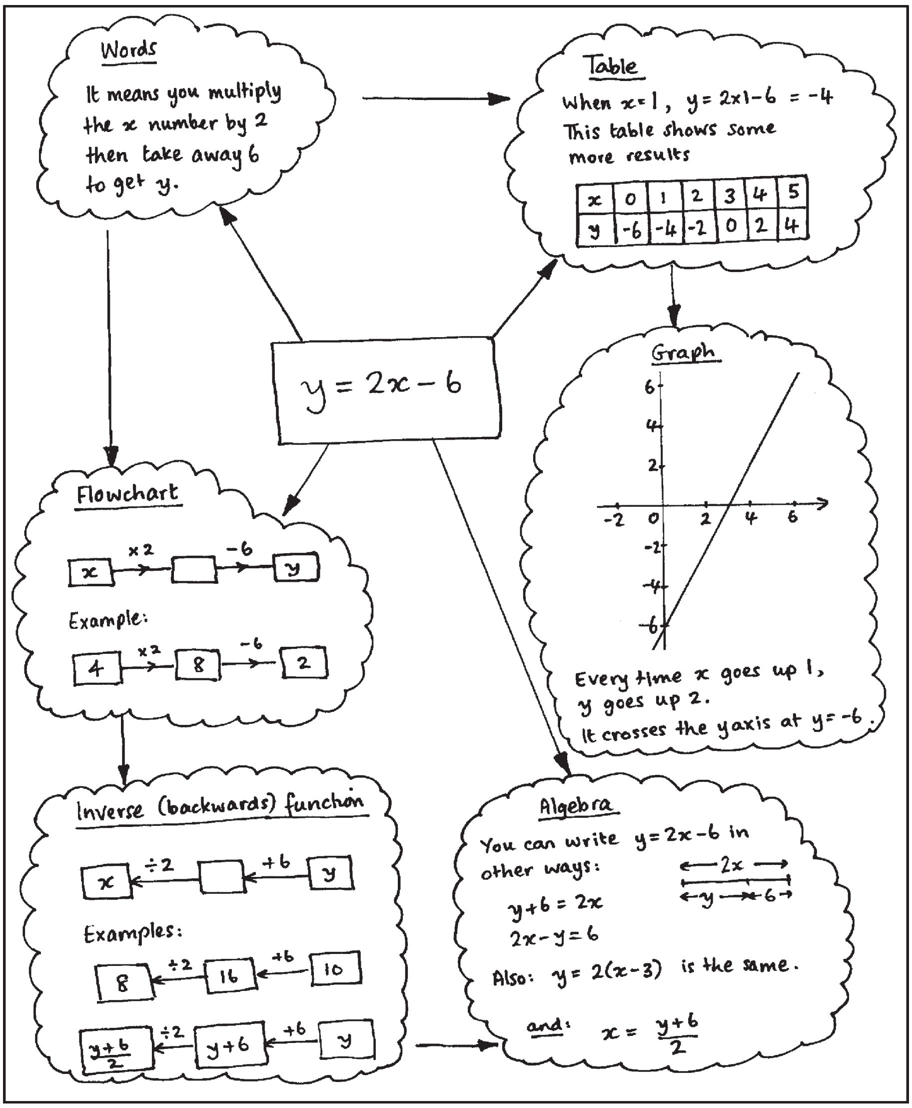
Examples of each poster go up on the classroom wall ensuring that every learner has something up. These serve as an excellent memory aid in later weeks."
Susan Wall
Wilberforce College



Perhaps the simplest way of using a poster is for learners to solve a problem collaboratively, explaining the thought processes involved at every step. Some teachers began by asking learners to divide the poster in two and then try to provide two different solutions, one in each column. Afterwards, posters were displayed and other learners were asked to comment on the solutions produced.

"Glue and coloured paper isn't just for kids."
AS learner
North Devon College

A second use of posters is to find out what learners already know about a given topic. For example, one teacher asked learners to write down all they knew about $y = 2x - 6$. Together, the following diagram was developed on the whiteboard. Learners were then given a variety of equations (the level of challenge was varied appropriately) and were asked to produce their own poster. The discussion enabled the teacher to assess how much learners knew about equations and how well they were able to link the ideas together.



It is important to emphasise that the aim of a poster is not to decorate the walls, but to encourage thinking. Learners find it stimulating to walk into a room where the walls are covered with creative thinking – even if it does contain errors.

3.4 Using card matching activities to focus on interpretation

These activities may be used as part of a longer session. There are many examples in this resource, and many more can be readily created for other topics. At their simplest, the cards may be dominoes or other shapes that fit together like jigsaws. They remove the need for written outputs and allow learners to focus all their attention on interpretation.

“These types of activity encourage learners to ‘have a go’ as they can move the cards around if they change their mind. It also encourages discussion as learners have to explain to each other why they think certain cards match.

Learners often prefer to ask each other if they do not understand and this gives an excellent opportunity for that. Learners often end up actually solving more problems than they would on a standard exercise but they do not realise this. It is also more motivating for learners who dislike written work or find it difficult to concentrate on written work.”
Susan Wall
Wilberforce College

The resource contains blank templates and photocopiable sheets to make the production of cards straightforward.

We have found that it is usually best if learners cut out the cards. This is more efficient from the teacher’s point of view, and learners begin to discuss the cards as they are doing this. We recognise, however, that this may not always be possible.

If the activity is interrupted by the end of the session, then it is not difficult to issue some paper clips so that learners can clip together the cards that they have sorted so far. A supply of small re-sealable polythene bags is also helpful for storing cards in between sessions.

Some teachers ask their learners to stick the cards onto large sheets of paper so that they can be displayed on the walls. This is not always essential, however.

4 • The types of activity

> "They are better behaved and motivated with these activities."
> **Inese Copeland**
> *Solihull College*

These resources are designed to develop mathematical thinking. We have attempted to do this through a number of different activity types. These types are not there to simply provide variety (though they do); they are devised to develop different ways of thinking.

4.1 **Classifying mathematical objects**
Learners devise their own classifications for mathematical objects, and apply classifications devised by others. They learn to discriminate carefully and recognise the properties of objects. They also develop mathematical language and definitions.

4.2 **Interpreting multiple representations**
Learners match cards showing different representations of the same mathematical idea. They draw links between different representations and develop new mental images for concepts.

4.3 **Evaluating mathematical statements**
Learners decide whether given statements are 'always true', 'sometimes true' or 'never true'. They are encouraged to develop rigorous mathematical arguments and justifications, and to devise examples and counterexamples to defend their reasoning.

4.4 **Creating problems**
Learners devise their own problems or problem variants for other learners to solve. This offers them the opportunity to be creative and 'own' problems. While others attempt to solve them, they take on the role of teacher and explainer. The 'doing' and 'undoing' processes of mathematics are vividly exemplified.

4.5 **Analysing reasoning and solutions**
Learners compare different methods for doing a problem, organise solutions and/or diagnose the causes of errors in solutions. They begin to recognise that there are alternative pathways through a problem, and develop their own chains of reasoning.

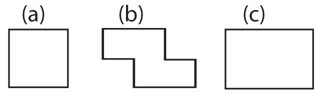
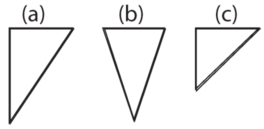
Each type of activity is described in more detail in the following pages.

4.1 Classifying mathematical objects

Mathematics is full of conceptual 'objects' such as numbers, shapes, and functions. In this type of activity, learners examine objects carefully, and classify them according to their different attributes. Learners have to select an object, discriminate between that object and other similar objects (what is the same and what is different?) and create and use categories to build definitions. This type of activity is therefore powerful in helping learners understand what is meant by different mathematical terms and symbols, and the process through which they are developed.

(i) Odd one out

Perhaps the simplest form of classification activity is to examine a set of three objects and identify, in turn, why each one might be considered the 'odd one out'. For example, in the triplets below, how can you justify each of (a), (b), (c) as the odd one out? Each time, try to produce a new example to match the 'odd one out'.

(a) (b) (c) 	(a) sin 60° (b) cos 60° (c) tan 60°
(a) a fraction (b) a decimal (c) a percentage	(a) $y = x^2 - 6x + 8$ (b) $y = x^2 - 6x + 9$ (c) $y = x^2 - 6x + 10$
(a) (b) (c) 	(a) 20, 14, 8, 2, . . . (b) 3, 7, 11, 15, . . . (c) 4, 8, 16, 32, . . .

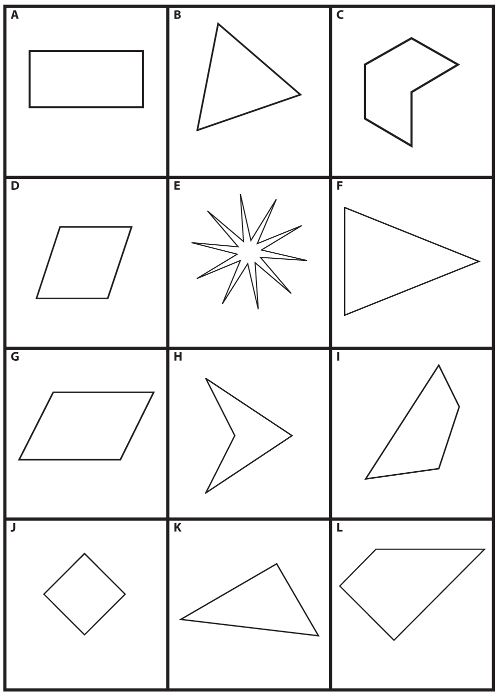
For example, in the first example, (a) may be considered the odd one as it has a different perimeter from the others, (b) may be considered the odd one because it is not a rectangle and (c) may be considered the odd one because it has a different area from the others.

(ii) Classifying using two-way tables

Typically, in these activities, learners are given a large collection of objects on cards and are asked to sort them into two sets according to criteria of their own choice. They then subdivide each set into two subsets using further criteria. They might then generate further objects for each set. Through discussing criteria, mathematical language is developed.

Learners are then given two-way grids on which they can classify the cards. Where they find that one cell of the grid is empty, they try to find an example that will fit, otherwise they try to explain why it is impossible. Some examples of the cards and grids are given below.

Shapes



	No rotational symmetry	Rotational symmetry
No lines of symmetry		
One or two lines of symmetry		
More than two lines of symmetry		

Quadratic functions

$y = x^2 + 2x + 4$	$y = x^2 - 5x + 4$
$y = 2x^2 - 5x - 3$	$y = x^2 - 4x + 4$
$y = x^2 + 7x - 3$	$y = 4 + 3x - x^2$
$y = x^2 + 5x - 2$	$y = 6x - x^2 - 9$
$y = x^2 - 3x - 1$	$y = x^2 + 10x + 9$
$y = x^2 + x + 3$	$y = x^2 + 4x + 4$
$y = x^2 - 2\sqrt{3}x + 3$	$y = 3x - x^2 + 7$

	Factorises with integers	Does not factorise with integers
Two x intercepts		
No x intercepts		
Two equal x intercepts		
Has a minimum point		
Has a maximum point		
y intercept is 4		

4.2 Interpreting multiple representations

> "The good thing about this was, instead of like working out of your textbook, you had to use your brain before you could go anywhere else with it. You had to actually sit down and think about it. And when you did think about it you had someone else to help you along if you couldn't figure it out for yourself, so if they understood it and you didn't they would help you out with it. If you were doing it out of a textbook you wouldn't get that help."
> **GCSE learner**
> *High Pavement College*

Mathematical concepts have many representations; words, diagrams, algebraic symbols, tables, graphs and so on. These activities are intended to allow these representations to be shared, interpreted, compared and grouped in ways that allow learners to construct meanings and links between the underlying concepts.

In most mathematics teaching and learning, a great deal of time is already spent on the technical skills needed to construct and manipulate representations. These include, for example, adding numbers, drawing graphs and manipulating formulae. While technical skills are necessary and important, this diet of practice must be balanced with activities that offer learners opportunities to reflect on their meaning. These activities provide this balance. Learners focus on interpreting rather than producing representations.

Perhaps the most basic and familiar activities in this category are those that require learners to match pairs of mathematical objects if they have an equivalent meaning. This may be done using domino-like activities. More complex activities may involve matching three or more representations of the same object.

Interpreting division notation

If I share 5 pizzas among 4 people, how much pizza will each get?

$5\overline{)4}$

0.8

$4 \div 5$

1.25

$\frac{5}{4}$

$5 \div 4$

$4\overline{)5}$

$\frac{4}{5}$

If I share 4 pizzas among 5 people, how much pizza will each get?

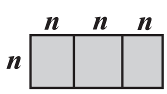
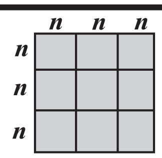
Typical examples might involve matching:

- times and measures expressed in various forms (e.g. 24-hour clock times and 12-hour clock times);
- number operations (e.g. notations for division – see below);
- numbers and diagrams (e.g. decimals, fractions, number lines, areas);
- algebraic expressions (e.g. words, symbols, area diagrams – see below);
- statistical diagrams (e.g. frequency tables, cumulative frequency curves).

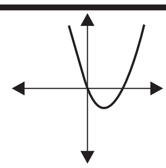
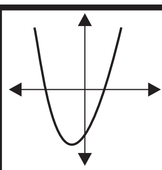
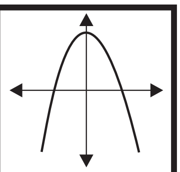
The discussion of misconceptions is also encouraged if carefully designed distracters are also included.

The examples below show some possible sets of cards for matching. They show how learners' attention can be focused on the way notation is interpreted, and common difficulties may be revealed for discussion.

Interpreting algebraic notation

	
Square n then multiply your answer by 3	Multiply n by 3 then square your answer
$9n^2$	$(3n)^2$
$3n^2$	Square n then multiply your answer by 9

Interpreting calculus notation

		
$f(x) = 16 - x^2$	$f(0) = 0$	$f(0) > 0$
$f(x) = x^2 + 2x - 3$	$f'(0) = 0$	$f'(0) > 0$
$f(x) = x(x - 2)$	$f'(0) < 0$	$f(0) < 0$

The sets of cards used in the sessions are usually much larger than those shown here. They also contain blank cards so that learners are not able to complete them using elimination strategies. Learners are asked to construct the missing cards for themselves.

When using such card matching activities, we have found that learners often begin quickly and superficially, making many mistakes in the process. Some become ‘passengers’ and let others do all the work.

The teacher’s role is therefore to ensure that learners:

- take their time and do not rush through the task;
- take turns at matching cards, so that everyone participates;
- explain their reasoning and write reasons down;
- challenge each other when they disagree;
- find alternative ways to check answers (e.g. using calculators, finding areas in different ways, manipulating the functions);
- create further cards to show what they have learned.

Often, learners like to stick their cards onto a poster and write their reasoning around the cards. For example, they might write down how they know that $9n^2$ and $(3n)^2$ correspond to the same area in the cards shown on page 20. It is important to give all learners an equal opportunity to develop their written reasoning skills in this way. If a group does not share the written work out equally, additional opportunities for written reasoning need to be created, perhaps through short, individual assignments.

These card sets are powerful ways of encouraging learners to see mathematical ideas from a variety of perspectives and to link ideas together.

4.3 Evaluating mathematical statements

These activities offer learners a number of mathematical statements or generalisations. Learners are asked to decide whether the statements are ‘always’, ‘sometimes’ or ‘never’ true, and give explanations for their decisions. Explanations usually involve generating examples and counterexamples to support or refute the statements. In addition, learners may be invited to add conditions or otherwise revise the statements so they become ‘always true’.

This type of activity develops learners’ capacity to explain, convince and prove. The statements themselves can be couched in ways that force learners to confront common difficulties and misconceptions. Statements might be devised at any level of difficulty. They might concern, for example:

- the size of numbers ("numbers with more digits are greater in value");
- number operations ("multiplying makes numbers bigger");
- area and perimeter ("shapes with larger areas have larger perimeters");
- algebraic generalisations ("$2(n + 3) = 2n + 3$");
- enlargement ("if you double the lengths of the sides, you double the area");
- sequences ("if the sequence of terms tends to zero, the series converges");
- calculus ("continuous graphs are differentiable");

. . . and so on.

Below and on page 23 are some examples. In each case (except for the probability example), the statements may be classified as 'always true', 'sometimes true' or 'never true'. Learners may enjoy working together, arguing about the statements and showing their agreed reasoning on posters.

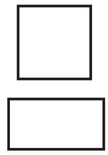
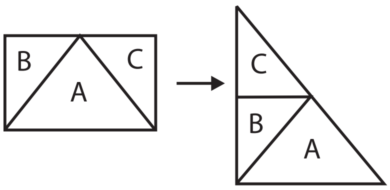
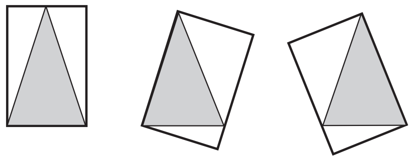
Throughout this process, the teacher's role is to:

- encourage learners to think more deeply, by suggesting that they try further examples ("Is this one still true for decimals or negative numbers?"; "What about when I take a bite out of a sandwich?"; "How does that change the perimeter and area?");
- challenge learners to provide more convincing reasons ("I can see a flaw in that argument"; "What happens when . . . ?");
- play 'devil's advocate' ("I think this is true because . . ."; "Can you convince me I am wrong?").

Number

If you divide a number by 2, the answer will be less than the number.	If you divide 10 by a number, your answer will be less than or equal to 10.
The square root of a number is less than or equal to the number.	The square of a number is greater than or equal to the number.

Perimeter and area

When you cut a piece off a shape, you reduce its area and perimeter. 	If a square and a rectangle have the same perimeter, the square has the smaller area. 
 When you cut a shape and rearrange the pieces, the area and perimeter stay the same.	 Draw a triangle. There are three ways of drawing a rectangle so that it passes through all three vertices and shares an edge with the triangle. The areas of the three rectangles are equal.

Equations, inequalities, identities

$p \div 12 = s + 12$	$3 + 2y = 5y$
$2t - 3 = 3 - 2t$	$q + 2 = q + 16$

Probability

In a lottery, the six numbers 3, 12, 26, 37, 44, 45 are more likely to come up than the six numbers 1, 2, 3, 4, 5, 6.	When two coins are tossed there are three possible outcomes: two heads, one head or no heads. The probability of two heads is therefore $\frac{1}{3}$.
There are three outcomes in a football match: win, lose or draw. The probability of winning is therefore $\frac{1}{3}$. 	In a true or false quiz, with 10 questions, you are certain to get 5 right if you just guess. 

4.4 Creating problems

In this type of activity, learners are given the task of devising their own mathematical problems. They try to devise problems that are both challenging and that they know they can solve correctly. Learners first solve their own problems and then challenge other learners to solve them. During this process, they offer support and act as ‘teachers’ when the problem solver becomes stuck.

Learners may be asked to construct their own problems for a variety of reasons. These include:

- enabling learners to reflect on their own level of attainment (e.g. “Make up some problems that test all the ways in which one might use Pythagoras’ theorem”);
- promoting an awareness of the range of problem types that are possible;
- focusing attention on the various features of a problem that influence its difficulty (e.g. size of numbers, structure, context);
- encouraging learners to consider appropriate contexts in which the mathematics may be used (e.g. create a range of problems about directed numbers using a money context);
- helping learners to gain ‘ownership’ over their mathematics and confidence when explaining to others.

At its most basic, this strategy may follow on from any exercise that the learners have been engaged in: “You’ve been working on these questions, now make up some more of your own for a neighbour to solve.” In this resource, however, creating problems has a more central role to play. The activities are mainly of two types (see pages 25 and 26).

(i) Exploring the 'doing' and 'undoing' processes in mathematics

In these situations, the poser creates a problem using one process, then the solver attempts to reverse that process in order to find a solution. In some cases the solution may not be the one expected, and this can create some useful discussion. In most situations, the poser has an easier task than the solver. This ensures that the task is solvable.

Exploring the 'doing' and 'undoing' processes in mathematics

<table>
<tr><th>Doing: The problem poser</th><th>Undoing: The problem solver</th></tr>
<tr><td>• calculates the area and perimeter of a rectangle (e.g. 5 cm × 7 cm).</td><td>• finds a rectangle with the given area (35 cm^2) and perimeter (24 cm).</td></tr>
<tr><td>• writes down an equation of the form $y = mx + c$ and plots a graph.</td><td>• tries to find an equation that fits the resulting graph.</td></tr>
<tr><td>• expands an expression such as $(x + 3)(x – 2)$.</td><td>• factorises the resulting expression: $x^2 + x – 6$.</td></tr>
<tr><td>• generates an equation step-by-step, starting with $x = 4$ and 'doing the same to both sides'.
<table>
<tr><td>times 10</td><td>$10x = 40$</td></tr>
<tr><td>Add 9</td><td>$10x+9 = 49$</td></tr>
<tr><td>Divide by 8</td><td>$\frac{10x+9}{8} = 6.125$</td></tr>
<tr><td>take 7</td><td>$\frac{10x+9}{8} - 7 = -0.875$</td></tr>
</table></td><td>• solves the resulting equation: $\frac{10x + 9}{8} - 7 = -0.875$</td></tr>
<tr><td>• writes down a polynomial and differentiates it.
$x^5 + 3x^2 – 5x + 2$</td><td>• integrates the resulting function.
$5x^4 + 6x – 5$</td></tr>
<tr><td>• writes down five numbers
2, 6, 7, 11, 14
and finds their mean, median, range.</td><td>• tries to find five numbers with the resulting values of mean = 8, median = 7 and range = 12.</td></tr>
</table>

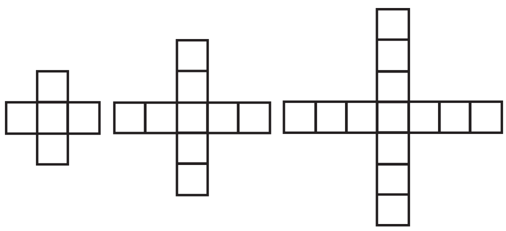
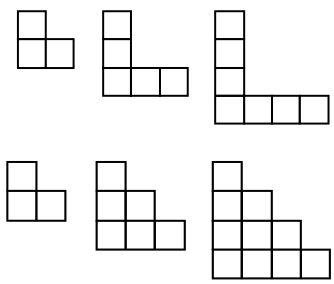
(ii) Creating variants of existing questions

It is helpful to do this in stages. Firstly, presented with a given question, ask “What other questions may have been asked?”. This helps learners to explore the structure of the situation more fully. Secondly, the learner tries to change the question in small ways. The numbers might be changed, for example. “What numbers make a solution impossible?” The diagram might be altered, and so on. Instead of just doing one question, the learner becomes aware that this question is just one example of a class of problems that might have been asked.

Throughout the process, the teacher’s role is to:

- explain and support the process of problem creation;
- encourage learners to support each other in solving the questions;
- challenge learners to explain why some problems appear to have several alternative solutions.

Creating variants of existing questions

Original exam question	**Possible revisions**
Some cross patterns are made of squares.  **Diagram 1** **5 squares** **Diagram 2** **9 squares** **Diagram 3** **13 squares** (a) How many squares will be in Diagram 6? (b) Write down an expression for the number of squares in Diagram n. (c) Which diagram will have 125 squares?	Write new questions for the original situation: • Can you have a diagram with 500 squares? How can you be sure? • The first cross is 3 squares long. How long is the nth cross? • The first diagram has a perimeter of 12. What is the perimeter of the 4th diagram? The 100th diagram? The nth diagram? • Is it possible to draw a cross diagram with a perimeter of 100? How can you be sure? Change the original situation: 

4.5 Analysing reasoning and solutions

The activities suggested here are designed to shift the emphasis from ‘getting the answer’ towards a situation where learners are able to evaluate and compare different forms of reasoning.

(i) Comparing different solution strategies

In many mathematics sessions, learners apply a single taught method to a variety of questions. It is comparatively rare to find sessions that aim to compare a range of methods for tackling a few problems. Many learners are left feeling that if they do not know ‘the right method’ then they cannot even begin to attempt a problem. Others are stuck with methods that, while generating correct answers, are inefficient and inflexible. These activities are designed to allow learners to compare and discuss alternative solution strategies to problems, thus increasing their confidence and flexibility in using mathematics. When ‘stuck’, they become more inclined to ‘have a go’ and try something. They thus become more powerful problem solvers.

In the following example, learners are asked to find as many different ways as they can of solving a simple proportion problem.

Paint prices

1 litre of paint costs £15.

What does 0.6 litres cost?

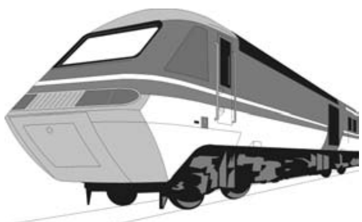
Chris:	It is just over a half, so it would be about £8.
Sam:	I would divide 15 by 0.6 You want a smaller answer.
Rani:	I would say one fifth of a litre is £3, so 0.6 litres will be three times as much, so £9.
Tim:	I would multiply 15 by 0.6
Teacher:	Do your methods give the same answers? If I change the 0.6 to a different number, say, 2.6, would your methods change? Why or why not? Does the method depend on the numbers?

(ii) Correcting mistakes in reasoning

These activities require learners to examine a complete solution and identify and correct errors. The activity may also invite the learner to write advice to the person who made the error. This puts the learner in a critical, advisory role. Often the errors that are exhibited are symptomatic of common misconceptions. In correcting these, therefore, learners have to confront and comment on alternative ways of thinking.

In the example below, four GCSE learners (Harriet, Andy, Sara and Dan) are discussing a common percentages misconception.

Rail prices



In January, fares went up by 20%.

In August, they went down by 20%.

Sue claims that:

"The fares are now back to what they were before the January increase."

Do you agree?

If not, what has she done wrong?

Harriet:	That's wrong, because . . .they went up by 20%, say you had £100 that's 5, no 10.
Andy:	Yes, £10 so its 90 quid, no 20% so that's £80. 20% of 100 is 80, . . . no 20.
Harriet:	Five twenties are in a hundred.
Dan:	Say the fare was 100 and it went up by 20%, that's 120.
Sara:	Then it went back down, so that's the same.
Harriet:	No, because 20% of 120 is more than 20% of 100. It will go down by more so it will be less. Are you with me?
Andy:	Would it go down by more?
Harriet:	Yes because 20% of 120 is more than 20% of 100.
Andy:	What is 20% of 120?
Dan:	96 . . .
Harriet:	It will go down more so it will be less than 100.
Dan:	It will go to 96.

(iii) Putting reasoning in order

Learners often find it difficult to produce an extended chain of reasoning. This process may be helped (or 'scaffolded') by offering the steps in the reasoning on cards, and then asking learners to correctly sequence the steps of the solution or argument. The focus of attention is thus on the underlying logic and structure of the solution rather than on its technical accuracy.

For example, in one activity, learners are given a collection of cards on which are separately written: three functions, their derivatives, second derivatives, factorisations of these, values of *x* that make these zero, and cards that show the turning points of the functions. In other words, learners are given a set of cards that contain every step in finding the turning points of three functions. Their task is to sort these cards into three sequences showing logical, step-by-step solutions (see, for example, Session C5 Card set A – *Stationary points*).

C5 Card set A – *Stationary points* (page 1)

$y = x^3 - 4x^2 + 5x + 11$	$y = x^3 - x^2 - x + 5$
$y = x^3 - 7x^2 - 5x + 9$	$3x^2 - 8x + 5 = 0$
$\frac{d^2y}{dx^2} = 6x - 8$	$\frac{dy}{dx} = 3x^2 - 14x - 5$
$x = -\frac{1}{3}, \ \frac{d^2y}{dx^2} = \ldots$	$x = \frac{5}{3}, \ x = 1$
$\frac{dy}{dx} = 3x^2 - 2x - 1$	$\frac{d^2y}{dx^2} = 6x - 2$
$x = 1, \ \frac{d^2y}{dx^2} = \ldots$	$x = -\frac{1}{3}, \ x = 5$
$(3x + 1)(x - 5) = 0$	$x = 1, \ \frac{d^2y}{dx^2} = \ldots$
$3x^2 - 14x - 5 = 0$	$3x^2 - 2x - 1 = 0$

C5 Card set A – *Stationary points* (page 2)

$(3x - 5)(x - 1) = 0$	$(3x + 1)(x - 1) = 0$
$x = 5, \ \frac{d^2y}{dx^2} = \ldots$	$\frac{dy}{dx} = 3x^2 - 8x + 5$
$x = -\frac{1}{3}, \ \frac{d^2y}{dx^2} = \ldots$	$x = -\frac{1}{3}, \ x = 1$
$\frac{d^2y}{dx^2} = 6x - 14$	$x = \frac{5}{3}, \ \frac{d^2y}{dx^2} = \ldots$
Maximum is at	Minimum is at ....
Minimum is at	Maximum is at ....
Maximum is at	Minimum is at ....

5 • Planning to use sessions

5.1 Introducing these activities to learners

Passive learning habits do not change overnight. When learners were first introduced to these activities, we found it important to explain their purpose and describe clearly how learners should work on them.

Some learners liked the activities from the start – they enjoyed interactive ways of learning. Others, however, appeared confused and challenged their teacher to explain why they were expected to learn in these new ways:

> "Why aren't we doing proper maths?"
>
> "Why do you want us to discuss?"
>
> "Why don't you just tell us how to do this?"

In order to explain the 'why?', some teachers found it helpful to draw a distinction between learning for fluency and learning for understanding:

> "Skills are things that you practise until you can do them almost without thinking. Think of the skill of using a computer keyboard, or playing the guitar. You practise regularly until you can type or play at speed without having to think about where your fingers are. If you neglect the practice you get rusty. In maths there are skills like knowing your tables, or knowing how to perform a calculation. These are things we need to practise until you can do them without thinking.
>
> Concepts and strategies are things that you need to learn to understand. If you want to understand a new idea or symbol or find different ways of doing a problem, you need to think about it and then talk about it with someone else. Often when you do this you learn to see something in a new way. When you really understand something you never forget it.
>
> In some sessions we will focus on practising skills. In others we will have discussions to consider ideas."

In order to explain the 'how?', some teachers find it helpful to set 'ground rules' and discuss these with learners. One possible handout is illustrated on page 31. There is also a short piece of footage on the DVD/video of learners talking about the approaches. This is specifically intended to be shown to other learners. From our experience, we find that it often takes several attempts at using discussion activities before learners really adapt to these new ways of working and begin to recognise the advantages.

A sample handout for learners

Discussing maths

Why discuss maths?

Many people think that there isn't much to discuss in maths. After all, answers are just right or wrong aren't they?

There is more to learning maths than getting answers. You need discussion in order to learn:

- what words and symbols mean;
- how ideas link across topics;
- why particular methods work;
- why something is wrong;
- how you can solve problems more effectively.

Teachers and trainers often say that they understand maths better when they start teaching it. In the same way, you will find that, as you begin to explain your ideas, you will understand them better.

As you begin to understand maths, you will remember it more easily and, when you do forget something, you will be able to work it out for yourself.

Some don'ts

- **Don't rush**
  It is more important to get a better understanding than to finish the activity.
- **Don't be a passenger**
  Don't let someone in your group 'take over'.

Stick to these basic rules and you will find that:

- you begin to enjoy maths more;
- you learn more from others;
- you find that your difficulties are the same as those experienced by others;
- you can help others too.

Some dos

- **Talk one at a time**
  Give everyone a chance to speak. Take it in turns to put forward ideas, explanations and comments. Let people finish.
- **Share ideas and listen to each other**
  If you don't understand what someone has said, keep asking 'why?' until you do understand. Ask them to give an example, draw a diagram or write down their explanation.
- **Make sure people listen to you**
  If you have just said something and are not sure if people understood you, ask them to repeat what you have just said in their own words.
- **Follow on**
  Try to say something that follows on from what the last person said.
- **Challenge**
  If you disagree with what people say, then challenge them to explain. Then put your point of view.
- **Respect each other's opinions**
  Don't laugh at other people's contributions (unless they're meant to be funny).
- **Enjoy mistakes**
  Don't worry about making mistakes. If you don't make mistakes, you cannot learn anything. It is sometimes interesting to make deliberate mistakes to see if your partner is listening.
- **Share responsibility**
  If the teacher asks your group to report back, make sure anyone in your group can do so.
- **Try to agree in the end**

5.2 Asking questions that make learners think

Teachers ask questions all the time. They serve a wide variety of purposes: to keep learners engaged during an explanation; to assess their understanding; to deepen their thinking or focus their attention on something. This process is often semi-automatic. Unfortunately, there are many common pitfalls. These include:

- asking questions with no apparent purpose;
- asking too many closed questions;
- asking several questions all at once;
- poor sequencing of questions;
- asking rhetorical questions;
- asking 'Guess what is in my head' questions;
- focusing on just a small number of learners;
- ignoring incorrect answers;
- not taking answers seriously.

In contrast, the research shows that effective questioning has the following characteristics.

- Questions are planned, well ramped in difficulty.
- Open questions predominate.
- A climate is created where learners feel safe.
- A 'no hands' approach is used, for example when all learners answer at once using mini-whiteboards, or when the teacher chooses who answers.
- Probing follow-up questions are prepared.
- There is a sufficient 'wait time' between asking and answering a question.
- Learners are encouraged to collaborate before answering.
- Learners are encouraged to ask their own questions.

There are many types of questioning that promote mathematical thinking. On these pages we can only offer a short 'taster' of what can be done. The table on page 33 opens up some of the possibilities. We have found that, as we begin to ask more thoughtful questions, then learners also begin to improve in the quality of the questions they ask themselves and each other. When planning each session, we found it very helpful to ask ourselves "What would be four really good questions to ask in this session?".

Some examples of open questions

Creating examples and special cases	Show me an example of . . .	• a square number. • an equation of a line that passes through (0, 3). • a shape with a small area and a large perimeter. • a real life problem where you have to calculate 3.4 ÷ 4.5
Evaluating and correcting	What is wrong with the statement? How can you correct it?	• When you multiply by 10 you add a nought. • $\frac{2}{10} + \frac{3}{10} = \frac{5}{20}$. • Squaring makes bigger. • If you double the radius you double the area.
Comparing and organising	What is the same and what is different about these objects?	• Square, trapezium, parallelogram. • An expression and an equation. • $(a + b)^2$ and $a^2 + b^2$. • $y = 3x$ and $y = 3x + 1$ as examples of straight lines. • $2x + 3 = 4x + 6$; $2x + 3 = 2x + 4$; $2x + 3 = x + 4$.
Modifying and changing	How can you change . . .	• this recurring decimal into a fraction? • this shape so that it has a line of symmetry? • the equation $y = 3x + 4$, so that it passes through (0, –1)? • Pythagoras' theorem so that it works for triangles that are not right-angled?
Generalising and conjecturing	This is a special case of . . . what? Is this always, sometimes or never true?	• 1, 4, 9, 16, 25, . . . • Pythagoras' theorem • The diagonals of a quadrilateral bisect each other. • $(3x)^2 = 3x^2$.
Explaining and justifying	Explain why . . . Give a reason why . . . How can we be sure that . . . Convince me that . . .	• $(a + b)(a – b) = a^2 – b^2$, by drawing a diagram. • a rectangle is a trapezium. • this pattern will always continue: $1 + 3 = 2^2$; $1 + 3 + 5 = 3^2$ . . . • if you unfold a rectangular envelope you will get a rhombus.

This table is derived from a more elaborate version in [11].

5.3 Exposing errors and misconceptions

Learners make mistakes for many reasons. They may just be due to lapses in concentration, hasty reasoning, or a failure to notice important features of a problem. Others, however, are symptoms of more profound difficulties.

> "Most learners have been positive with the increasing participation in discussion, creating questions etc. Some found some of the activities difficult – but this challenged them and they got a lot out of it. Most have gained confidence in articulating ideas. Some are still afraid to make mistakes however."
> **Frank Mills**
> *Chesterfield College*

There is now a vast body of research literature documenting learners' mistakes in mathematics (e.g. [10, 23]). This work shows that mistakes are often the result of consistent, alternative interpretations of mathematical ideas. These should not be dismissed as 'wrong thinking' as they are natural, sometimes necessary, stages of conceptual development. For example, most learners generalise from their early experiences that:

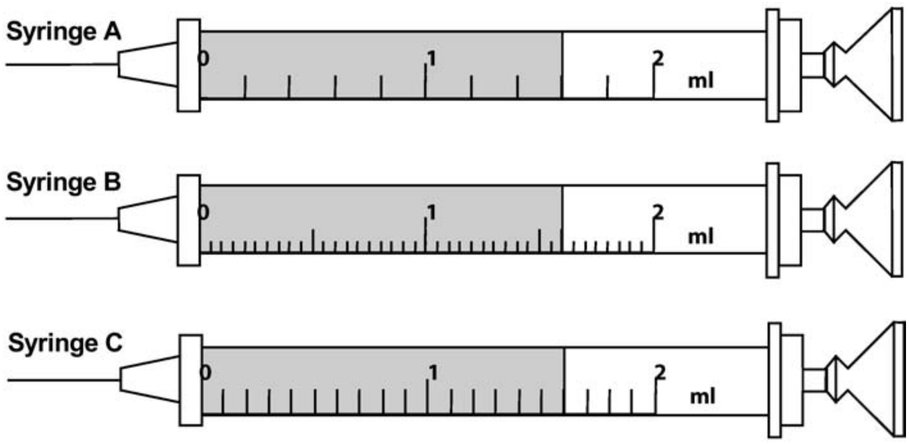
- "you can't divide smaller numbers by larger ones";
- "division always makes numbers smaller";
- "the more digits a number has, then the larger is its value";
- "shapes with bigger areas have bigger perimeters";
- "letters represent particular numbers";
- "'equals' means 'makes'".

There are two common ways of reacting to these misconceptions.

- Try to avoid them:
  "If I warn learners about the misconceptions as I teach, they are less likely to happen. Prevention is better than cure."
- Provoke them and use them as learning opportunities:
  "I actively encourage learners to make mistakes and learn from them".

The first reaction invokes a medical metaphor which is unhelpful. Misconceptions are not diseases that can be avoided by 'better teaching', neither are they 'caught' through a casual encounter. They are reasoned, alternative ways of thinking. Research suggests that teaching approaches which encourage the exploration of misconceptions through discussions result in deeper, longer term learning than approaches which try to avoid mistakes by explaining the 'right way' to see things from the start [5, 7, 22]. The research studies have taught us the following.

- It is helpful if discussions focus on known difficulties. Rather than posing many questions in one session, it is better to focus on a challenging question and encourage a variety of interpretations to emerge, so that learners can compare and evaluate their ideas.
- Questions can be juxtaposed in ways that create a tension (sometimes called a 'cognitive conflict') that needs resolving. Contradictions arising from conflicting methods or opinions can create an awareness that something needs to be learned. For example, asking learners to say how much medicine is in each of syringes A–C (below) may result in answers such as "1.3 ml, 1.12 ml and 1.6 ml". "But these quantities are all the same." This provides a start for a useful discussion on the denary nature of decimal notation.



- Activities should provide opportunities for meaningful feedback. This does not mean providing summative information, such as the number of correct or incorrect answers. More helpful feedback is provided when learners compare results obtained from alternative methods until they realise why they get different answers.
- Sessions should include time for whole group discussion in which new ideas and concepts are allowed to emerge. This requires sensitivity so that learners are encouraged to share tentative ideas in a non-threatening environment.
- Opportunities should be provided for learners to 'consolidate' what has been learned through the application of the newly constructed concept.

Many of the activities in this resource are designed with the intention to provoke such discussions.

There are several ways of creating the conditions in which learners can feel able to discuss common mistakes and misconceptions without feeling threatened. Two suggestions are given below.

- Ask learners to solve a problem in pairs. Collect in some examples of different responses and write these on the board anonymously. Add a few extra ones that illustrate other interesting discussion points that you have already prepared. Then ask learners to discuss and debate these responses.
- Give learners a completed past examination paper to mark. This should illustrate interesting and significant errors that show underlying misconceptions. It is often good to include responses that the learners themselves have made, but rewritten so that they become anonymous. Learners usually seem to enjoy the role shift when they are asked to become examiners. Encourage them to write comments indicating the source of the error and helpful advice to the 'candidate'.

5.4 Managing small group discussion

As we have already noted, there is now general agreement in research that cooperative small group work has positive effects on both social skills and mathematics learning, but this is dependent on shared goals for the group and individual accountability for the attainment of the group [3].

Group work may not always be appropriate. When the purpose of the session is to develop fluency in a particular skill and there is little to discuss, then individual practice may be more suitable. This should not constitute the whole diet, however. Collaborative group work is necessary when the purpose of the session is to develop conceptual understanding or strategies for solving more challenging problems. In these cases, learners need to share alternative views, interpretations or approaches.

"During the lessons, the students have been motivated and have been eager to participate. There were loads of discussions. Even the students who were quiet and shy came out of their shells and discussed."
Ranvir Singh Lally
John Willmott School

There is a clear difference between working **in** a group and working **as** a group. It is quite common to see learners working independently, even when they are sitting together. 'Disputational talk', in which learners simply disagree and go on to make individual decisions, is not beneficial. Neither is 'cumulative' talk in which learners build uncritically on what each other has said. For true collaborative work, learners need to develop 'exploratory talk' consisting of critical and constructive exchanges, where challenges are justified and alternative ideas are offered [12, 13]. It is not enough for learners to simply give each other right answers, as this does not produce enhanced understanding [14]. The most helpful talk appears to be that where the participants work on and elaborate each other's reasoning in a collaborative rather than competitive atmosphere. Exploratory talk enables reasoning to become audible and publicly accountable.

Learning gains are related to the degree to which learners give help to, or seek help from, each other. This is most likely to happen when the groups are 'near mixed ability' that is middle attainers with high attainers, or middle attainers with low attainers [3]. A number of research studies have found that, while group work is positively related to achievement when it is respectful and inclusive, it is negatively related to achievement when the interactions are disrespectful and unequal [14].

The research indicates that group work may be difficult when learners do not have adequate sharing skills, participation skills, listening and communication skills. These involve:

- allowing others in the group adequate opportunity to express their ideas, and being patient when they have difficulty doing so;
- overcoming shyness and being willing to cooperate with the group;
- listening rather than simply waiting to offer one's own point of view;
- taking time to explain and re-explain until others understand.

The evidence and feedback from our own work suggests that learners take time to learn to work in these ways but, when they do so, the benefits are considerable.

Some of the questions that teachers frequently asked were:

- How should I group my learners? Should I use pairs or larger groups?
- How can I stop more dominant learners from taking over?
- How can I ensure that all learners contribute?
- When and how should I intervene?

"It has taught me not to talk so much. The materials create the time to listen."
Derek Robinson
Bishop Luffa School

There are no definitive answers to these questions. It depends on the nature of your particular learners and the circumstances in which you teach them. However, here are some findings from our experience that may be of help.

(i) Organising groups

As regards group size, most of the teachers found that asking learners to work in pairs or threes was most effective. Some tried larger groups but felt that this encouraged 'passengers'.

Many teachers allowed learners to choose their own partners to work with. Others asked particular individuals to sit together, perhaps because they had contrasting views or because the teacher knew that one might be able to help the other in particular ways. It was found helpful, however, to vary group compositions from time to time.

When problem solving, or when trying to reach agreement over an interpretation, teachers found it helpful to change the group size as the session developed. A 'snowball' approach was popular with some. In this, learners began by responding to a task individually. This ensured that everyone had something to contribute to the group discussions. Then pairs were formed and learners were asked to try and reach agreement. Finally, pairs joined together so that a broader consensus could be reached.

When groups had completed a task, teachers sometimes asked a representative from each group to share ideas with a different group. This approach often created surprise when groups had worked differently on the same task.

"It was more fun and a better way to understand the work. Much better than just listening."
GCSE learner
Derby College

Some teachers were surprised to observe learners who usually find the subject difficult forcefully putting their ideas to more confident learners and helping them to see alternative ways of doing things. The group dynamics appeared to be more dependent on the personalities and relationships between learners than on their competence with mathematics.

(ii) The teacher's role

During small group discussions, teachers often appear unsure of their own role. Should they 'hang back' and let the discussions go their own way, or should they intervene? We found the following list of dos and don'ts particularly helpful.

Small group discussion: the teacher's role

- **Make the purpose of the task clear**
  Explain what the task is and how learners should work on it. Also, explain why they should work in this way. "Don't rush, take your time. The answers are not the focus here. It's the reasons for those answers that are important. You don't have to finish, but you do have to be able to explain something to the whole group."

- **Keep reinforcing the 'ground rules'**
  Try to ensure that learners remember the ground rules that were discussed at the beginning, using a checklist such as that shown on page 31. Encourage learners to develop responsibility for each other's understanding. "I will pick one of you to explain this to the whole group later – so make sure all of you understand it."

- **Listen before intervening**
  When approaching a group, stand back and listen to the discussion before intervening. It is all too easy to interrupt a group with a predetermined agenda, diverting their attention from the ideas they are discussing. This is not only annoying and disruptive (for the group), it also prevents learners from concentrating.

- **Join in; don't judge**
  Try to join in as an equal member of the group rather than as an authority figure. When teachers adopt judgmental roles, learners tend to try to 'guess what's in the teacher's head' rather than try to think for themselves: "Do you want us to say what we think, or what we think you want us to say?".

- **Ask learners to describe, explain and interpret**
  The purpose of an intervention is to increase the depth of reflective thought. Challenge learners to describe what they are doing (quite easy), to interpret something ("Can you say what that means?") or to explain something ("Can you show us why you say that?").

- **Do not do the thinking for learners**
  Many learners are experts at making their teachers do the work. They know that, if they 'play dumb' long enough, then the teacher will eventually take over. Try not to fall for this. If a learner says that they cannot explain something, ask another learner in the group to explain, or ask the learner to choose some part of the problem that they **can** explain. Don't let them off the hook. When a learner asks the teacher a question, don't answer it (at least not straight away). Ask someone else in the group to answer.

- **Don't be afraid of leaving discussions unresolved**
  Some teachers like to resolve discussions before they leave the group. When the teacher leads the group to the answer, then leaves, the discussion has ended. Learners are left with nothing to think about, or they go on to a different problem. It is often better to reawaken interest with a further interesting question that builds on the discussion and then leave the group to discuss it alone. Return some minutes later to find out what has been decided.

5.5 Managing whole group discussion

Whole group discussions usually take place after small group discussions, towards the end of a session. This need not always be the case, however. Sometimes it is helpful to have a whole group review at the beginning of a session. Alternatively, something significant may arise during a period of small group discussion that requires the attention of everyone.

Whole group discussions have a variety of purposes.

- **Presenting and reporting.** Learners may be asked to describe something they have done, an answer they have obtained and their method for obtaining it, or to explain something they have learned. Their ideas may be compared and evaluated by the whole group.
- **Recognising and valuing.** Some of the ideas generated in the discussion will be more important and significant than others. It is the teacher's role to recognise these 'big ideas', make them the focus of attention and give them status and value.
- **Generalising and linking.** This involves showing how the ideas generated in the session may be developed and used in other situations. Learning is thus put into a wider context.

"One year on, I have found that these materials have changed the way I teach. My teaching is much less judgmental. I don't agree or disagree with an answer but ask for comments, and refuse to accept a sloppy explanation. This causes no end of annoyance when the initial answer is correct but they think that because I am seeking clarification it must be wrong. In the 'best case' scenario, one student may take a deep breath and produce a lucid explanation in response to the poor explanations offered by other students."
Joan Ashley
Cambridge Regional College

The teacher's role depends to some extent on which purpose is being served. If the role is one in which learners are reporting back, then the teacher may decide to adopt a managerial role. For example, some teachers preferred to stay at the back of the room and encourage representatives from each group to go to the front and present the group's ideas. This typically entailed showing posters or working through methods on the board. This was followed by a period when the presenting group was asked questions by other learners with the teacher acting as the 'chair'.

If the purpose is to recognise and value particular learning points in the work, then the teacher will have a more proactive role. Typically, the teacher will use prepared questions on the significant learning points that have arisen. This could also involve the use of mini-whiteboards.

If the purpose is to generalise and link ideas, then the teacher may introduce a new problem, or an extension to a problem that has already been considered by the whole group and, again through questioning, show how the ideas from the session may be applied to this new problem.

The suggestions on page 41 have been found useful in promoting useful and lively discussions in which learners feel able to exchange and examine ideas and hypotheses [21].

The teacher and whole group discussions

The suggestions below have been found useful in promoting discussions in which learners feel able to exchange ideas. Learners are usually only able to participate if they have done some preliminary talking about the issues in pairs or small groups first. Remember to allow time for this.

Mainly be a 'Chairperson' or 'Facilitator' who:

- directs the flow of the discussion and gives everyone a chance to participate;
- does not interrupt or allow others to interrupt the speaker;
- values everyone's opinion and does not push his or her point of view;
- helps learners to clarify their own ideas in their own words.

"Listen to what Jane is saying."
"Thanks, Serena, now what do you think, Hannah?"
"How do you react to that, Tom?"
"Are there any other ideas?"
"Could you repeat that please, Ali?"

Occasionally be a 'Questioner' or 'Challenger' who:

- introduces a new idea when the discussion is flagging;
- follows up a point of view;
- plays devil's advocate;
- focuses on an important concept;
- asks provocative questions, but not 'leading', or 'closed' questions.

"What would happen if…?"
"What can you say about the point where the graph crosses the axis?"

Don't be a 'Judge' or 'Evaluator' who:

- assesses every response with a 'yes', 'good' or 'interesting', etc. This tends to prevent others from contributing alternative ideas, and encourages externally acceptable performances rather than exploratory dialogue;
- sums up prematurely.

"That's not quite what I had in mind.""
"You're nearly there."
"Yes, that's right."
"No, you should have said …"
"Can anyone see what's wrong with Kwanele's answer?"

5.6 Meeting the needs of all learners

It may be inconvenient, but it is an inescapable fact: all learners are different. There are many strategies that teachers use in order to try to ensure that every learner is given a challenge appropriate to their attainment and preferred ways of learning. The following four approaches are quite common.

(i) Differentiate by quantity

This strategy assumes that higher attaining learners will work more quickly and extra work should be held 'up one's sleeve' to cater for this. To us, however, 'more work' is unhelpful when this only means 'more of the same'. These learners need to explore ideas in more depth, not merely cover more ground.

(ii) Differentiate by task

In this approach, learners are given different problems or activities, according to their learning needs. This approach is difficult to implement well, because it presumes that the teacher can prejudge the attainment of each learner accurately and that there is also a bank of suitable problems or activities that may be drawn on. During trials of our resources, some teachers decided in advance that some learners would not be able to cope with particular concepts and ideas and, when using 'card matching' activities, they removed all the cards that might be 'too difficult'. This was unsatisfactory as it denied learners the opportunity even to engage with these ideas. Teachers who used the activities 'as written', however, were often surprised when learners showed that they were able to learn and discuss even quite complex mathematical ideas. We would not recommend, therefore, that the activities be 'simplified' before giving them to learners.

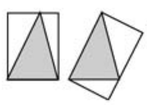
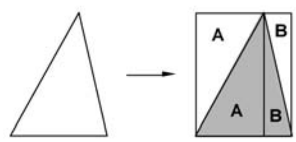
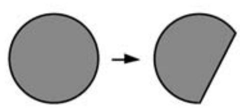
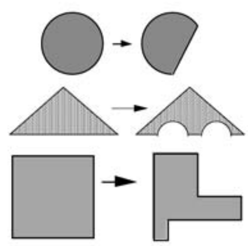
A second approach is to give learners some choice in the activities they undertake. For example, in one group, learners were asked to choose between a straightforward, a challenging and a very challenging task. Few chose the straightforward task; most preferred a challenge. This approach assumes that learners are able to make a realistic assessment of their own ability to solve the problem. It works less well with the less confident.

(iii) Differentiate by level of support

In this strategy, all learners are given the same task, but are offered different levels of support, depending on the needs that become apparent.

This avoids the danger of prejudging learners and so is used in some of the activities in this resource.

For example, in the session **SS4 Evaluating length and area statements**, learners are asked to decide whether given statements are always, sometimes or never true. If they struggle with this, then 'hints' cards may be given out to provide further help without giving too much away. Of course, carefully chosen hints may be given orally during any activity.

Statement cards	Hints cards
 Draw a triangle. There are three ways of drawing a rectangle so that it passes through all three vertices and shares an edge with the triangle. The areas of the three rectangles are equal.	 What fraction of each rectangle is the triangle? What happens when the triangle contains an obtuse angle?
 When you cut a piece off a shape you reduce its area and perimeter.	What happens to the area and perimeter with these cuts? 

Card matching activities differentiate by allowing learners to take many different approaches. Learners who prefer visual images may decide to begin with cards that show diagrams, while those who prefer verbal representations may decide to begin with cards showing words. Learners who are finding the topic difficult may be given additional cards that show more accessible representations, while higher attaining learners may be asked to construct additional, more complex examples.

(iv) Differentiate by outcome

Open activities that encourage a variety of possible outcomes offer learners the opportunity to set themselves appropriate challenges. This approach is used in many of the activities. For example, some activities invite learners to create their own classifications or their own problems and examples. Teachers may encourage learners to 'make up questions that are difficult, but that you know you can get right'. In one GCSE retake group, for example, the following equations (and solutions) were constructed by learners during the activity: **A2 Creating and solving equations**. Other learners were then asked to try to solve them. The teacher was surprised both by the complexity of the examples generated and by the enthusiasm of the learners.

Sally	$\dfrac{3\left(\frac{x^2}{8}\right) - 0.375}{2} = 15$	Chris	$\dfrac{2x-4}{2} + 2 = 7$
Fazal	$\dfrac{2x-15}{11} + 2 = 7$	Rachel	$40 - \left(\dfrac{r^2-10}{5}\right) = -3$
James	$(5x+5)^2 = 2025$	Pete	$3\left(\dfrac{x^2+4}{5}\right) - 6 = 18$

5.7 Using formative assessment

As Ofsted says:
In failing colleges . . .
"A common feature of a number of unsatisfactory lessons was the failure of teachers to make regular checks on students' learning and their determination to continue with the planned work even when the students clearly did not understand it." [17]

In succeeding colleges . . .
"Teachers do not plan in a vacuum, but on the basis of a detailed knowledge of their learners' prior attainment and potential, acquired through initial assessment and induction and recorded in individual learning plans. Lesson plans become an active means of orchestrating the sequence of the proposed activities, according to the needs of the individuals within the group, but with sufficient built-in flexibility to be able to respond to the unexpected, should it occur. Differentiated approaches are planned for all students, not just the less able." [18]

In mathematics, learners are frequently tested, marks awarded and records produced, but this does little to promote learning. Effective assessment practices, however, enable teachers to build on learners' prior knowledge, and match their teaching to the needs of each learner. The research evidence offers the following advice on how to do this [8, 9, 25].

(i) Plan assessment opportunities

As teachers, we assess all the time, through questioning and 'eavesdropping' on the work of learners during sessions. These 'on the hoof' assessments, however, need to be supplemented by systematic, planned assessments. As well as informing teachers, planned assessment should also help learners become more aware of **what** they still need to learn and **how** they might go about learning these things.

(ii) Assess groups as well as individuals

Given the usual size of classes, formative assessment can be very time consuming, particularly when it is focused on providing detailed, formative feedback to individuals. Group activities, such as those supplied in this resource, however, allow opportunities to observe, listen, and question groups of learners in ways that provide a wealth of formative assessment evidence that may be used to refocus teaching. Sessions in which posters are produced are particularly helpful group assessment opportunities.

(iii) Encourage self-assessment and peer-assessment

Studies on formative assessment point clearly to the value of learners assessing themselves. Through this process learners become aware of what they need to know, what they do know, and what needs to be done to narrow the gap. One way of achieving this is to give copies of learning objectives to learners, ask them to produce evidence that they can achieve these objectives and, where they cannot, discuss what they need to do next.

Over time, it is also possible to foster a collaborative culture in which learners take some responsibility for the learning of their peers. This involves making time for learners to read through each other's work and to comment on how it may be improved.

(iv) Use a range of more divergent assessment techniques

It is possible to distinguish between two types of assessment – convergent and divergent [25].

- **Convergent assessment** ("Can you do this ...?"). This approach is often characterised by tick lists and can-do statements. The teacher asks closed questions in order to ascertain whether or not the learner knows, understands or can do a predetermined thing. This is the type of assessment most used in written tests.
- **Divergent assessment** ("Show me what you know about ..."). The teacher asks open questions that allow learners opportunities to describe and explain what they know, understand or can do. The outcome is not predetermined.

Divergent strategies are particularly useful for formative purposes. While using the activities in this resource, teachers have used a variety of divergent assessment strategies. Learners have, for example, been asked to:

- respond to sets of open questions using mini-whiteboards;
- produce posters to summarise what they know about a given topic or alternative approaches to solving a given problem;
- produce short revision guides for a topic;
- interview each other, in pairs, about what they have learned;
- correct and comment on work produced by other learners.

(v) Give feedback that is useful to learners

Evidence suggests that the only type of feedback that promotes learning is a meaningful comment (**not** a numerical score) on the quality of the work and constructive advice on how it should be improved. Indeed marks and grades usually detract learners from paying attention to qualitative advice. The research evidence [9] clearly shows that helpful feedback:

- focuses on the task, not on marks or grades;
- is detailed rather than general;
- explains **why** something is right or wrong;
- is related to objectives;
- makes clear what has been achieved and what has not;
- suggests what the learner may do next;
- describes strategies for improvement.

This doesn't necessarily mean writing long comments at the bottom of each piece of work. It is helpful to give comments orally and then perhaps ask learners to summarise what has been said in writing.

(vi) Change teaching to take account of assessment

As well as providing feedback to learners, good assessment feeds forward into teaching. This sounds obvious but it is hard to do well. It means that we must try to become more flexible and responsive teachers, losing some of the 'must cover' anxiety that arises from the zealous use of syllabus frameworks [25]. The aim should be for learners to learn, not for us to teach.

Pages 48 and 49 show one account of a formative assessment event with a GCSE group [6]. The teacher in this case asked his learners to create their own test and mark scheme, use it on each other and analyse the results. This took three hours. It not only helped learners to revise a topic, it also made them aware of some important examination strategies.

Making up a test

Step 1: Reviewing the work and devising the test

The teacher began by telling his learners that he wanted them to work in groups and write questions based on the functions work they had recently completed. He then said that, when they had finished writing their questions, he would select the best ones and make up a test from them. They would then do this test during their next session. The teacher led a short discussion on the content to be covered and on 'what makes a good question'. Learners worked together to prepare their questions, without reference to textbooks. They reviewed the content covered while they did this.

Step 2: Doing the test

The teacher arrived for the second session with copies of a test paper made up of one question from each group. Learners first scanned the test, eagerly searching for 'their own' questions, then settled down to completing it. At one stage during the test, the learner who had written question 5 remarked:

> "I've done it wrong. In question 5, it should be 39 not 9."

The other learners groaned as they altered the incorrect numbers on the test paper. By the end of the session they had completed the test and the teacher collected their scripts.

Step 3: Marking responses and data handling

In the third session, the teacher discussed with learners how mark schemes are created. In particular he talked through the notion that marks are given both for method and for accuracy. This was new to some learners, who hadn't realised the importance of showing working. They also discussed the relative demands of the questions and whether some are worth more marks than others. It was eventually decided to allocate 50 marks for the eight questions.

The teacher then passed round four completed tests. "If one of them belongs to you, then don't tell anybody." Groups then marked each paper using its own mark scheme.

The teacher collected together on the board the marks allocated to each paper by each group. Learners then discussed the range of marks for the test and each question, decided who the lenient and harsh markers were, and so on. This revised some data handling knowledge.

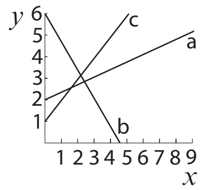
Finally, the teacher asked the whole group what had been learned.

> "How different people's minds work differently."
>
> "How difficult it is to mark work."
>
> "It's not just the mark that's important."
>
> "If you write your work out clearly you're more likely to get more marks."

The learners' test (left) and one learner's mark scheme (below) (including errors)

The Test

1. Find the equation for lines a, b and c.



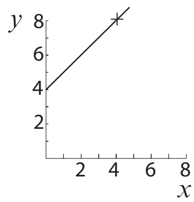
2. In general, how do you find the equation from the graph?
3. Find the relationship in these sets of numbers and add another pair to the set:

A	1	2	4	6	10
B	1	7	19	31	55

4. Describe in not more than 50 words the definition of inverse.
5. Find the relationship:

C	–1	5	–5	0	2
D	3	9	–21	9	21

6. Given this rule, $4x - 2$, describe the numbers which go with (a) 3, (b) 6, (c) 10.
7. Find the equation of this line



8. Give an example of
   (a) a positive gradient
   (b) a negative gradient
   (c) a zero gradient
   Draw lines on the same set of axes.

Answer sheet

1a) $x = \frac{y}{x} = \frac{1}{3}$ ① ①   Total 6

$y = 2$ ①

$y = \frac{1}{3}x + 2$ ③

b) $y = -6$ ① $\frac{y}{x} =$ ① ①   Total 6

$y = 1{\cdot}3x - 6$ ③

c) $x = \frac{y}{x} = 1$ ① ①

$y = 1$ ①

$y = 1x + 1$ ③

② You find an equation from the graph by first drawing a right angled triangle on the line. Then you measure the distance of both lines and ÷ the vertical by the horizontal. This equals the x part of your answer. To find the intercept you see where the line crosses the y axis. You put the numbers together to get a formula.

Total 5

③ A 1 2 4 6 10 ①

B 1 7 19 31 55 ①

$y = \times 6 - 5$ ① ①

Total = 4

④ Inverse means opposite. When an equation is an inverse of another it is reversed so the inverse of $a \times 3 = b$ is $b \div 3 = a$ ① ①

Total 2

⑤ C –1 5 –5 0 2 22 ①

D 3 39 –21 9 21 ①

$y = \times 6 + 9$ ① ①

Total = 3

⑥ The rule is $4x - 2$

a) $3 \times 4 - 2 = 10$

b) $6 \times 4 - 2 = 22$

c) $10 \times 4 - 2 = 38$

Total = 4

⑦ Gradient $= \frac{4}{4} = 4$ ②

y intercept $= 4$ ①

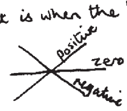
equation $y = 4x + 4$ ③

Total = 6

⑧ a) A positive gradient is when the y numbers are all above 0 and the line is going up ②

b) A negative gradient is when some of the y numbers are below 0 and the line goes down ②

c) A zero gradient is when the line is in a straight line and is horizontal ②



②

Total marks = 50

5.8 The role of the computer

Technology offers us many exciting new resources that engage and motivate learners to work on mathematics. Computers, data projectors and interactive whiteboards open up new ways to enhance the learning process. Examples of these are included in this resource.

There are many reasons for using computers as an integral part of the learning programme.

- They are **interactive**. They enable learners to explore situations by changing something on the screen and observing the effect.
- They provide **instant feedback**. Learners can immediately see the consequences of decisions they make. This makes them very useful for formative assessment.
- They are **dynamic**. Learners are able to visualise concepts in new ways. For example, they allow graphs or geometrical objects to be generated and transformed.
- They link the learner with **the real world**. For example, real data may be downloaded and used in sessions.

> "Generally, the evidence tells us that learners are developing increasingly sophisticated ICT skills. They are also more likely to use ICT at home than in their institution and be motivated in their learning when ICT is used. This is especially true of older school pupils and learners studying at higher levels in FE colleges. Similarly, there are indications of general demand for the use of ICT resources among FE students and a perception that ICT is currently being under-utilised in learning and teaching.
>
> Learners are ready to embrace higher levels of use of ICT and are increasingly coming to expect it as a routine aspect of learning and teaching." [4]

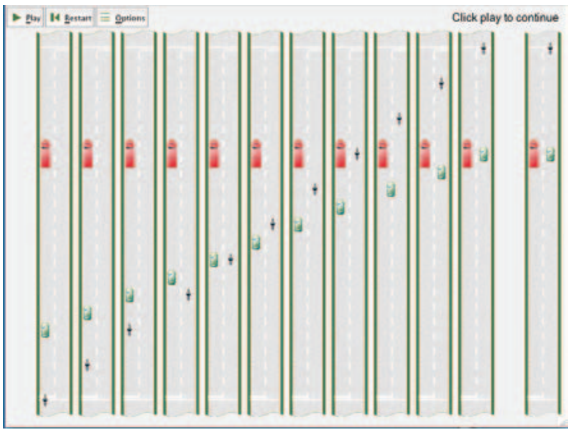
The range of available software continues to expand rapidly. Broadly speaking educational software is of three types:

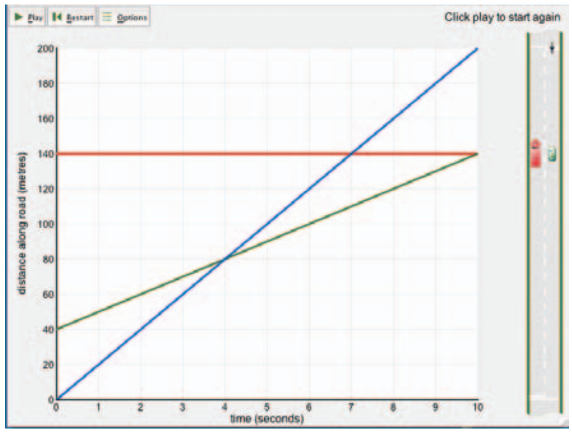
- **Generic applications.** These include spreadsheets and databases that were not written specifically for mathematics teachers, but which allow new mathematical problems to be posed, and new ways of working on problems to be explored.
- **Mathematics packages.** These include software packages designed for graph drawing, dynamic geometry, data handling and algebraic manipulation. These provide us with powerful, dynamic, interactive ways of developing mathematical concepts. They are essential resources for any mathematics department, but they have many features and take some time to learn.
- **Purpose-built applets.** These are simple computer programs that are designed to perform a single task and which are run from within a larger application. Often these are available over the internet and run in a browser window. They are very easy to pick up and use with almost no preparation.

In this project, we began by selecting and using applets simply because they are so easy to use. In the session outlines, we show how we integrated these into our teaching in ways that encouraged discussion and reflection. On pages 52 to 55 we give a very brief introduction to each of the programs we developed and used for this resource.

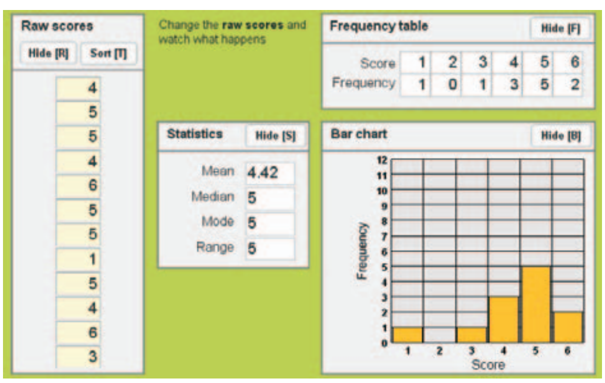
Three of the programs are designed to help learners **interpret mathematical representations**.

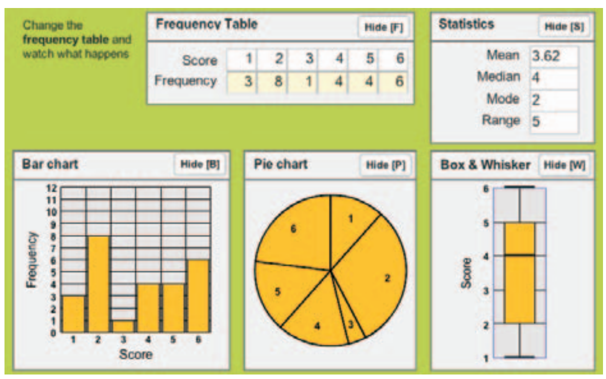
Traffic is a simple introduction to distance-time graphs. It shows an animation of traffic moving along a road, as viewed from the air. Photographs are taken of this road at one-second intervals and are laid side-by-side. The patterns produced are then clearly linked to the distance-time graph of the motion.



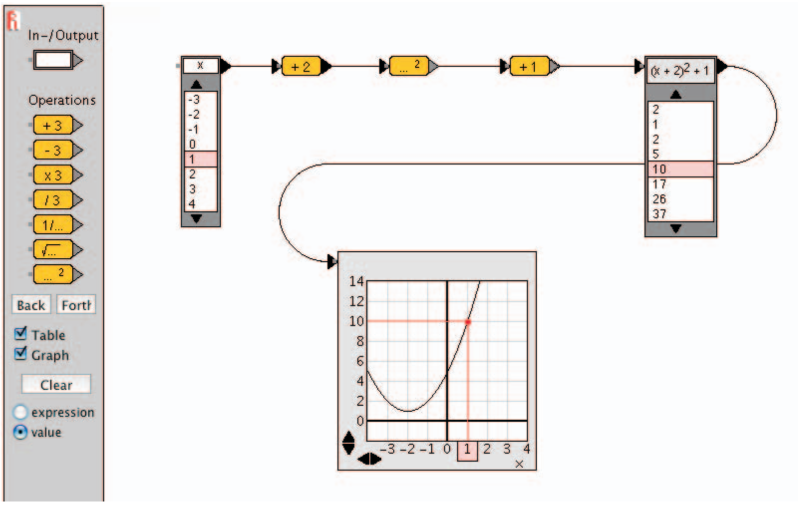


Statistics shows a number of different representations of a given set of statistical data. As the data are varied, one can explore what happens to each representation.



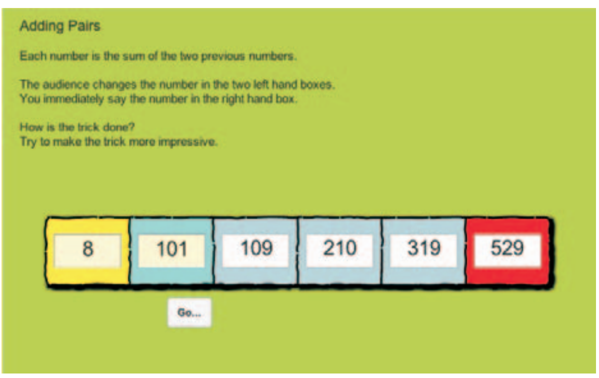


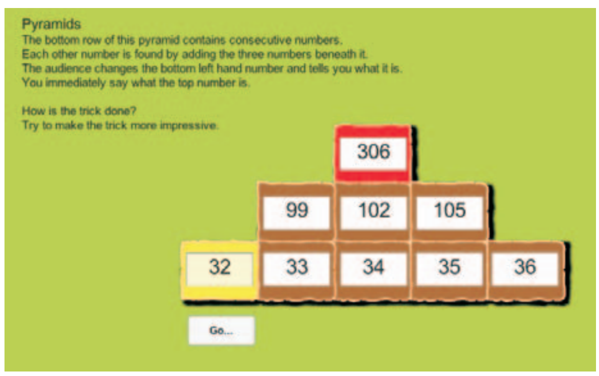
Machines allows learners to construct function machines and observe the tables and graphs that they generate. The program may also be used to construct inverse functions and solve simultaneous equations.



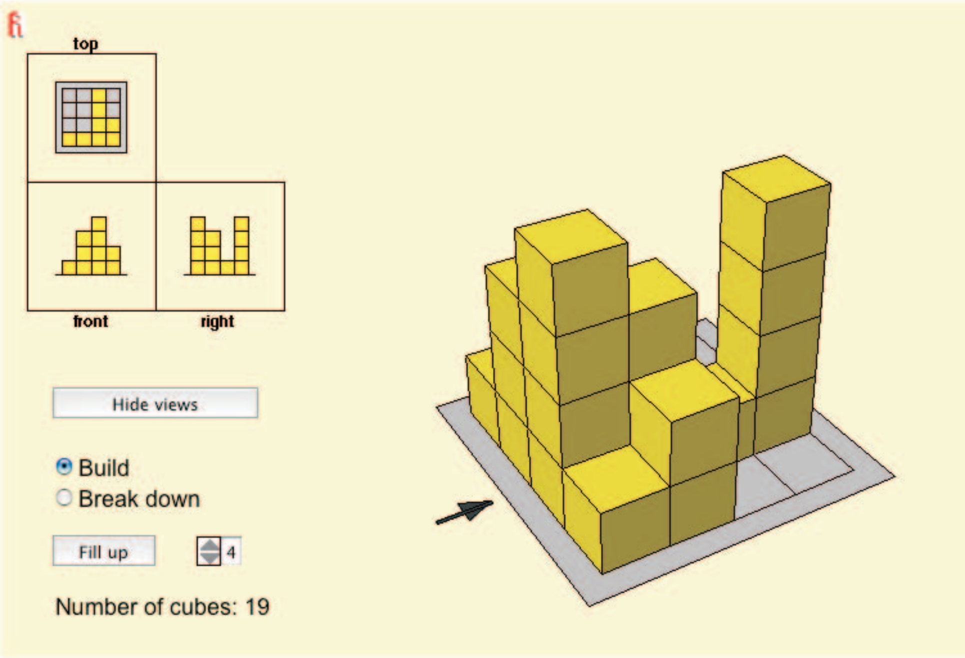
Three of the programs offer **situations to explore**.

Number magic encourages learners to explore number ‘tricks’, explain why they work using algebra and generate their own, harder versions.

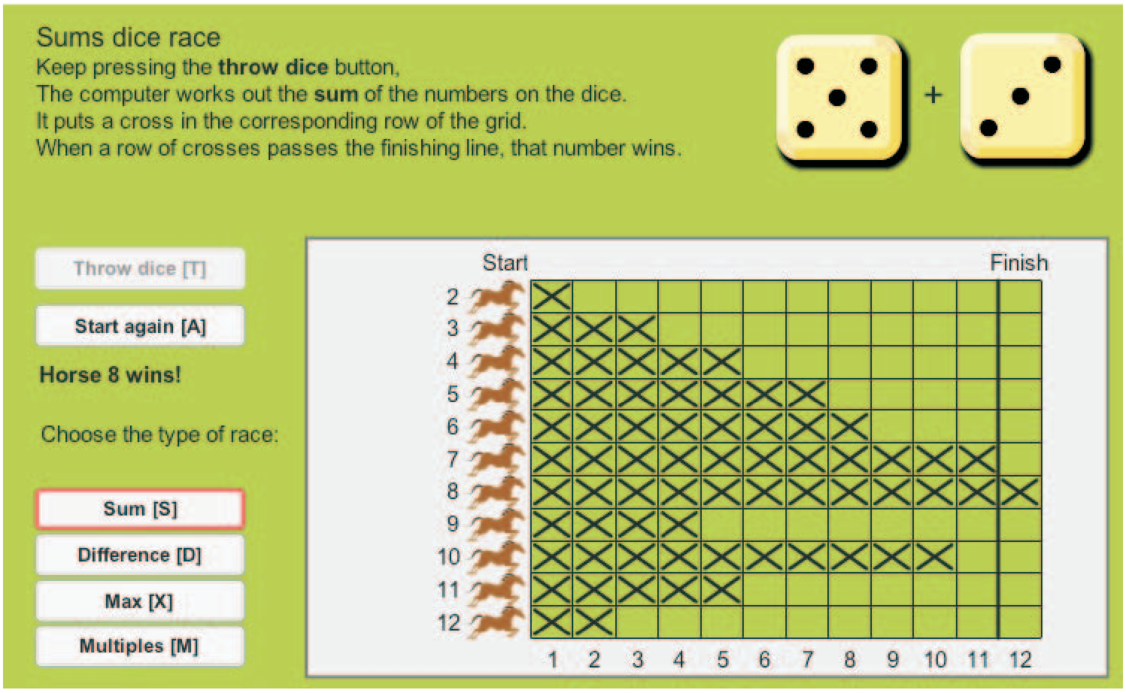




Building houses challenges learners to explore connections between 3-dimensional models and their plans and elevations.

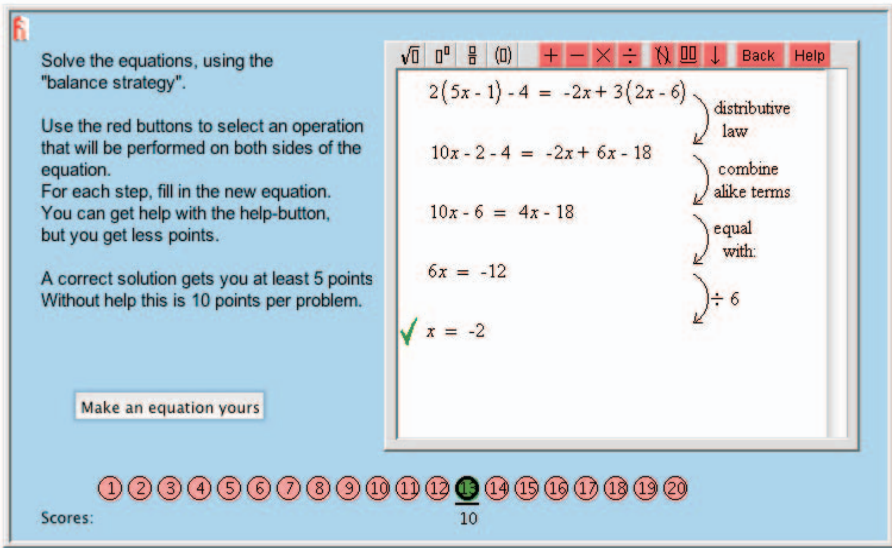


Coin and dice races encourages learners to make statistical predictions, carry out experiments, generate data and explain the patterns observed.

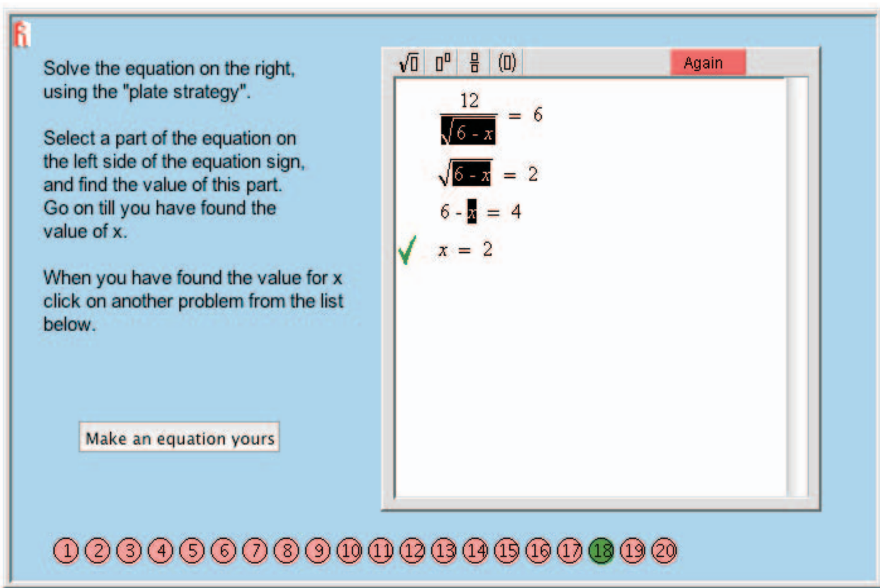


Finally, there are two programs that offer learners individual practice at creating and solving equations using two different strategies. These are called **Balance** and **Cover-up**.

Balance



Cover-up



This small selection of software provides a taste of the software that is currently being developed in mathematics education. Many of these may be found on the internet. ‘Machines’, ‘Building houses’, ‘Balance’ and ‘Cover-up’ are examples of a very helpful collection of resources being developed in Holland by the Freudenthal Institute and made available at the website www.fi.uu.nl. They are used here with kind permission.

6 • Some frequently asked questions

How do the activities fit into a scheme of work?

There are two main types of schemes of work.

- **Target-defined** schemes provide a list of content (usually the exam syllabus) together with dates by which each part must be 'covered'. Often, there are cross-references to textbooks, past papers and other resources.
- **Activity-defined** schemes provide an organised list of learning activities and problems, cross-referenced to learning objectives and dates.

There are several potential difficulties with target-defined schemes.

- They tend to predetermine the pace of learning.
- They ignore the prior knowledge and attainment of learners.
- The focus is on what is covered – not what is learned.
- They do not encourage subject content to be linked together.

In this project, several providers have begun to rewrite their schemes of work so that they are organised by activities. These schemes are seen as working documents that change as the activities are improved and revised. They are flexible in that they allow teachers to change the course of their work according to the needs of the learners. For example, in the course of using an activity, a teacher will find that learners achieve some (possibly unanticipated) learning outcomes, while others may not be achieved. The teacher can now select the next activity from the scheme so that it focuses on those unachieved needs.

Clearly, the activities that constitute these resources will fit well into such a scheme.

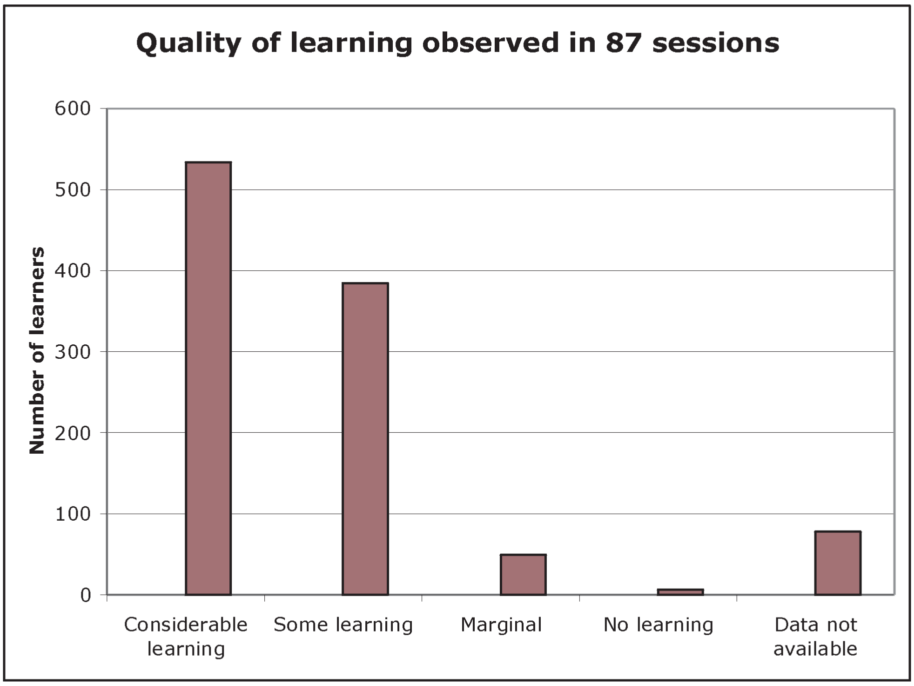
> "Now I'm sure that when I move onto the next topic they do have a proper understanding of the topic that they've just covered. The students have got a much better understanding of new topics and this is much better than me just carrying on regardless of whether anybody actually understands anything so that I can get through the syllabus by the end. I think it's better that they know 60 percent of 70 percent than 10 percent of 100 percent."
> **Jane Annets**
> *Tower Hamlets College*

Discussion takes so long – will we cover the syllabus?

Teachers often use the term 'cover' to refer to the facts and skills that they have demonstrated and explained to their learners. It is of course true that discussion does take time, and learners will encounter the ideas more slowly than if they are 'lectured'. However, what is 'covered' does not correspond to the learning that has taken place. When one considers the longer-term benefits of discussion, one finds that the learning is more permanent (it is understood rather than imitated) and learning gains are real rather than imagined.

What will the response be from learners?

During the development of the early drafts of these resources, we observed 87 sessions. During each session, we estimated the quality of learning that was taking place. This was done by talking to learners and observing their discussions and work, then estimating the number of learners that we would allocate to each category: considerable learning, some learning, marginal learning, no learning. The results of observing 1 052 learners are shown here. This is, of course, a subjective judgment.

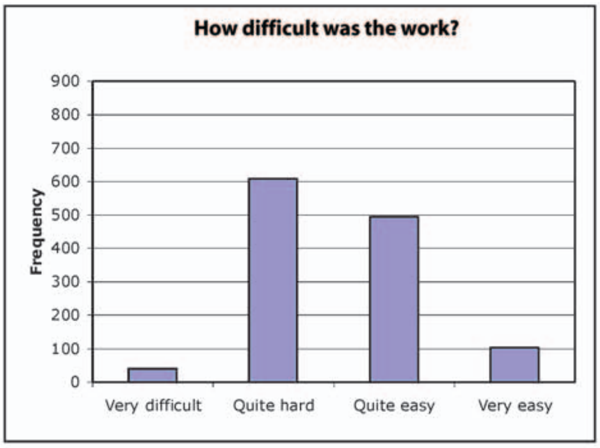


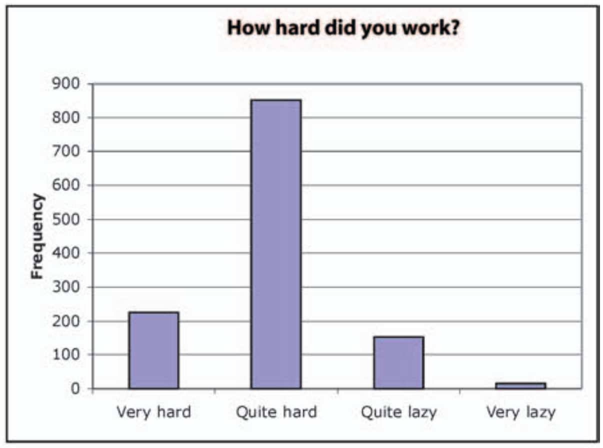
We also asked 1 247 learners to respond to questionnaires regarding the difficulty of the work that was chosen for them, the degree to which they engaged with it, their enjoyment of the work and their own perceptions of learning. The results are shown on page 58. Learners felt that they worked hard in the sessions, enjoyed the work and learned quite a lot.

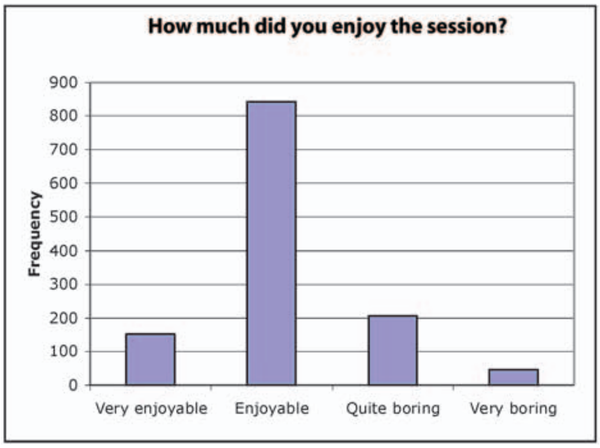
What will Ofsted say?

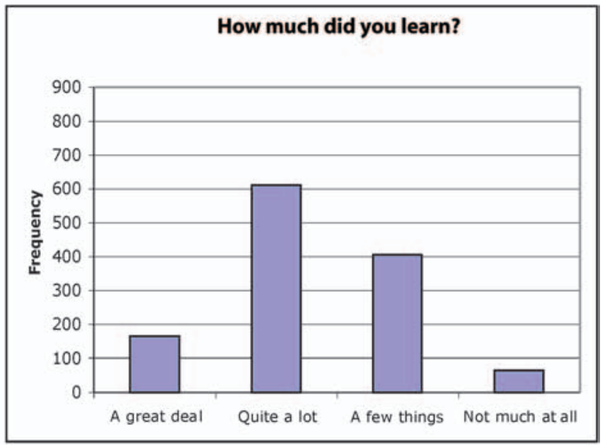
The practices we have outlined in this resource are consistent with advice and guidance produced by Ofsted.

> "A vital element in good teaching is getting the right balance between giving information and setting challenging work which engages students and promotes the acquisition of new knowledge, skills and understanding . . . Keep in mind the following characteristics of effective teaching and learning in which teachers:









- enjoy their subject and confidently and energetically present it to students in ways which capture their interest;
- check their understanding of mathematical ideas, revise and refine different techniques and approaches;
- regularly encourage students to discuss mathematics and explain what they are doing, challenge them to find alternative/shorter/more elegant methods, and monitor knowledge and understanding through well-chosen questions;
- stimulate students to think mathematically, to look beyond routines and outcomes, to ask questions and to search for reasons why something works;
- make effective use of available resources . . . Calculators, software . . . and practical resources." [15]

During the course of the project several of the colleges were inspected by Ofsted while using the resources. When the resources have been used in the ways intended, the outcomes have been very positive. For example, here are just two extracts from Ofsted reports that have singled out learning by collaboration and discussion in mathematics.

"There is much good teaching, particularly in mathematics where innovative approaches encourage participation and increase motivation and understanding. In almost all lessons, a wide range of carefully planned activities enthuse the students and encourage them to share ideas. Students' contributions are valued and a problem solving approach is used with an emphasis on developing confidence and mathematical skills. Students work together collaboratively and are reaching high standards. In a very effective GCSE mathematics lesson, a practical activity was used that involved matching formulae to written descriptions. The students said that the activity-based approach helped to make them retain their learning and their levels of participation and enjoyment were very high. In AS-level and GCE A2 lessons, students enjoy using individual whiteboards for their rough calculations and for offering solutions to questions posed by the teacher. In one outstanding lesson, the whiteboards were used to support an effective question and answer session that was very skilfully managed by the teacher and moved at a lively pace. The students were relaxed and confident as they discussed their solutions and explained concepts to each other; they then worked very productively in small groups producing posters that explained the concepts and included annotated worked examples. The interactive whiteboard was used very effectively by the teacher to recall earlier solutions and develop ideas." [19]

"Teaching and learning in mathematics are very good. Teachers use very effective strategies in mathematics lessons, . . . The active learning approach to teaching is transforming and revitalising learning. Lesson planning and schemes of work in science and mathematics are good, providing a sound basis for teaching and learning. In the best lessons, a wide range of learning activities challenge all students. Often, there is exceptional use of ICT in lessons. Students are well motivated and ask questions which display interest and understanding." [20]

Some suggestions for further reading

Books

Bills C., Bills L., Mason J. and Watson A., *Thinkers: a collection of activities to provoke mathematical thinking*, 2004, Association of Teachers of Mathematics.

Boaler J., *Experiencing school mathematics: teaching styles, sex and setting*, 1997, Open University Press.

Brown S.I. and Walter M.I. *The art of problem posing*, 1990, Lawrence Erlbaum Associates.

Gates P. (ed.), *Issues in mathematics teaching*, 2001, RoutledgeFalmer.

Haggerty L. (ed.), *Teaching mathematics in secondary schools: a reader*, 2002, RoutledgeFalmer.

Muijs D. and Reynolds D., *Effective teaching: evidence and practice*, 2001, Paul Chapman Publications.

Ollerton M. and Watson A., *Inclusive mathematics 11–18*, 2001, Continuum International Publishing.

Orton A. and Wain G. (eds), *Issues in mathematics teaching*, 1994, Continuum International Publishing.

Prestage S. and Perks P., *Adapting and extending secondary mathematics activities: new tasks for old*, 2001, David Fulton Publishers.

Swan M. and Green M., *Learning mathematics through discussion and reflection*, 2002, Learning and Skills Development Agency.

Watson A., De Geest E. and Prestage S., *Deep progress in mathematics*, 2003, University of Oxford.

Periodicals

Mathematics in Schools, Mathematical Association.

Mathematics Teaching, Association of Teachers of Mathematics.

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[14] Muijs D. and Reynolds D., *Effective teaching: evidence and practice*, 2001, Paul Chapman Publications.

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[16] Ofsted, 'Mathematics in secondary schools', in *Ofsted Subject Reports*, London, 2002, HMSO.

[17] Ofsted, *Why colleges fail*, London, 2004a, HMSO.

[18] Ofsted, *Why colleges succeed*, London, 2004b, HMSO.

[19] Ofsted, *Wilberforce College inspection report*, London, 2004c, HMSO.

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[21] Swan M. (ed.), *The language of functions and graphs: an examination module*, 1985, The Joint Matriculation Board/Shell Centre for Mathematical Education.

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